Decision Framework Based on Binary Intersections and Fuzzy-Soft Set Integration

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Abstract

This study introduces a novel decision-making framework, termed Dual-Consensus Decision for Decision Making (DCD-DM), that unifies Local Union of Binary Intersections (LUBI) and Global External Intersection (GEI) strategies to support robust consensus-based reasoning. The model is further extended through the incorporation of soft-fuzzy set theory, enabling decision mechanisms to account for uncertainty and partial agreement among criteria. Stepwise formulations are presented to demonstrate how weighted aggregation and union-based logic can refine decision outcomes in complex environments. The proposed approach is benchmarked through illustrative examples, revealing its capability to balance strict consensus and flexible compromise. This framework opens pathways for scalable and intelligent multi-criteria decision systems in uncertain domains.

Keywords: Dual-Consensus Decision Making, Soft-Fuzzy Sets, Binary Intersection, and-product, LUBI, GEI, Multi-Criteria Decision Making, Uncertainty Modelling.

1. Introduction

The complexity of modern systems and decision-making criteria in ambiguous environments has led to the emergence of mathematical tools capable of dealing with uncertainty. Soft set theory [1], proposed by D. Molodtsov in 1999, is at the forefront of robust mathematical frameworks for representing uncertain knowledge. It has proven effective in addressing the shortcomings of other theories dealing with uncertainty, such as probability theory and fuzzy and rough set theories [2, 3]. Besides the theoretical developments of soft sets that have been extensively studied in the literature [4-9] researchers have developed several soft set-based methodologies to solve decision-making problems as in [6,10-12]. Traditional soft set models offer flexibility by allowing decisions to be parameterized. Decision making in multi-criteria environments often demands a balance between local agreements among parameters and global consensus across decision makers. Classical decision-making approaches may fail to accommodate partial agreement, uncertainty, or conflicting evaluations, especially when strict intersection-based criteria are applied. Which often produces empty decision sets, or when dealing with binary and fuzzy information simultaneously. This creates a need for a more flexible and expressive decision model capable of integrating both hard and soft consensus mechanisms.

This paper proposes a novel decision-making framework that combines binary logic and soft-fuzzy set theory to address challenges in multi-criteria environments characterized by uncertainty and partial agreement. Specifically, it introduces a Dual-Consensus Decision Making (DCD-DM) model that integrates the Local Union of Binary Intersections (LUBI) and the Global External Intersection (GEI) to capture both localized and global agreement among evaluators. The framework is extended with fuzzy-soft logic to enable graded outputs and confidence-based ranking, enhancing decision robustness in complex scenarios.

The paper is organized as follows: Preliminaries on soft sets, fuzzy sets, and soft-fuzzy theory are introduced in section 2. Section 3 presents the core dual-consensus algorithm and its underlying mathematical structure. Additionally, a comparative example is provided to demonstrate the practical implementation of the model and evaluate its effectiveness in comparison with previous methods. Section 4 explores the main mathematical properties of the DCD operator, including symmetry and De Morgan's laws, with formal proofs. Section 5 presents special cases of the DCD operator and introduces a weighted and extension to handle cases of strict or partial consensus. The section includes techniques for generating more inclusive decisions. Section 6 explores a fuzzy-soft scoring mechanism for ranking decisions. Finally, section 7 concludes the paper and outlines directions for future research.

2. Preliminaries

To understand the mechanism of the proposed algorithm, it is first necessary to clarify some basic concepts on which the adopted methodology is based.

2.1 Soft Sets [1]

Let U be a universal set and E a set of parameters. A soft set over U is a pair (F, A), or simply F_A, where $A \subseteq E$ and $F: A \longrightarrow P(U)$. For each parameter $e \in A$, the mapping F assigns a subset $F(e) \subseteq U$.

2.2 Fuzzy Sets [2]

A fuzzy set μ over U is defined as $\mu: U \longrightarrow [0,1]$, assigning a degree of membership to each element.

2.3 Soft Fuzzy Sets [13]

Let U be a universal set, E a set of parameters, and $A \subseteq E$. A pair (F, A) is called a soft fuzzy set over U if:

i. There exists a fuzzy relation

$$\mu_F: A \times U \rightarrow [0,1],$$

represent degrees of satisfaction for each element $h \in U$ with respect to each element $e \in A$.

ii. The mapping $F: A \rightarrow P(U)$ defined by via the α-cut of μ_F i.e.,

$$F(e) = \{h \in U: \mu_F(e,h) \geq \alpha, \alpha \in [0,1]\}.$$

2.4 And-Product [6]

Let (F, A) and (G, B) be two soft sets defined over a universe U with corresponding parameter sets E. Then and-product of (F, A) and (G, B) denoted by $(F, A) \wedge (G, B) = (H, A \times B)$, where:

$$H(x, y) = F(x) \cap G(y) \; \forall \, (x, y) \in A \times B.$$

3. Dual-Consensus Decision (DCD-DM) Model

This section introduces the Dual Consensus Decision (DCD) method, which builds on the "and-product" operation to support intersection-based decision-making. The and-product is used to generate pairwise intersections that underpin the Local Union of Binary Intersections (LUBI) and the Global External Intersection (GEI), enabling the identification of shared and distinct elements across decision matrices and forming the core of the DCD framework.

3.1 Basic Definitions

Definition 3.1 Local Union of Binary Intersections ($\mathbf{LUBI}_{int}$)

Let (F, A) and (G, B) be two non-null soft sets over the universe U, with approximate functions $F: A \rightarrow P(U)$ and $G: B \rightarrow P(U)$, respectively. Then, the Local Union of Binary Intersections is defined as the union of all pairwise intersections $F(x) \cap G(y)$ for parameters x and y in the shared set $A \cap B$. Mathematically, this can be expressed as:

$$\text{LUBI}_{int} = \bigcup_{x,y \in A \cap B} \big(F(x) \cap G(y)\big).$$

Definition 3.2 Global External Intersection ($\mathbf{GEI}_{int}$)

Let (F, A) and (G, B) are two non-null soft sets over the universe U, with approximate functions $F: A \rightarrow P(U)$ and $G: B \rightarrow P(U)$ respectively. Then, the Global External Intersection is defined as the union of all pairwise intersections $F(x) \cap G(y)$ for parameters $x \in A$ and $y \in B$, where x and y are not shared between A and B (i.e., $x, y \notin A \cap B$). Mathematically, this can be expressed as:

$$\text{GEI}_{int} = \bigcup_{\substack{x \in A, y \in B \\ x,y \notin A \cap B}} \big(F(x) \cap G(y)\big).$$

3.2 Dual Consensus Decision (DCD-DM)

By using the definitions Local Union of Binary Intersections and Global External Intersection, a Dual Consensus Decision method is constructed by the following algorithm:

Step 1: Input the soft sets (F, A) and (G, B).

Step 2: Find and-product of (F, A) and (G, B).

Step 3: Find a Local Union of Binary Intersections - LUBI_{int}.

Step 4: Find Global External Intersection – GEI_{int}.

Step 5: Dual Consensus Decision (DCD-DM)

The final decision is computed by intersecting LUBI_{int} and GEI_{int} i.e.,

$$\text{DCD}_{int} = \bigcup_{x,y \in A \cap B} \big(F(x) \cap G(y)\big) \cap \bigcup_{\substack{x \in A, y \in B \\ x,y \notin A \cap B}} \big(F(x) \cap G(y)\big).$$

To illustrate the applicability of the proposed model, we consider a previously studied example (example 4 in [6]). Reusing this example allows for a comparison between the proposed dual-consensus framework and prior decision-making techniques, as it enables the evaluation of results under identical conditions.

Example 1: In this example the universal set $U = \{h_1, h_2, h_3, \dots, h_{48}\}$, represents the full set of candidates under consideration. The evaluation parameters $E = \{e_1, e_2, e_3, \dots, e_7\}$, correspond to the parameters used by decision-makers to assess candidates. Two evaluators select subsets of these parameters, denoted by $A = \{e_1, e_2, e_4, e_7\}$ and $B = \{e_1, e_2, e_5\}$, respectively.

The soft sets are constructed, and there and-product are then computed

$$(F, A) = \left\{ \begin{matrix} (e_1, \{h_4, h_7, h_{13}, h_{21}, h_{28}, h_{31}, h_{32}, h_{36}, h_{39}, h_{41}, h_{43}, h_{44}, h_{49}\}), \\ (e_2, \{h_1, h_3, h_{13}, h_{18}, h_{19}, h_{21}, h_{22}, h_{24}, h_{28}, h_{32}, h_{36}, h_{42}, h_{44}, h_{46}\}), \\ (e_4, \{h_2, h_3, h_{13}, h_{15}, h_{18}, h_{23}, h_{25}, h_{28}, h_{30}, h_{33}, h_{36}, h_{38}, h_{42}, h_{43}\}), \\ (e_7, \{h_1, h_5, h_{12}, h_{13}, h_{17}, h_{20}, h_{24}, h_{28}, h_{29}, h_{34}, h_{36}, h_{41}, h_{45}, h_{47}\}) \end{matrix} \right\}$$

$$(G, B) = \left\{ \begin{matrix} (e_1, \{h_3, h_4, h_5, h_8, h_{14}, h_{21}, h_{22}, h_{26}, h_{27}, h_{34}, h_{35}, h_{37}, h_{40}, h_{42}, h_{46}\}), \\ (e_2, \{h_1, h_4, h_7, h_{10}, h_{11}, h_{13}, h_{15}, h_{21}, h_{29}, h_{30}, h_{32}, h_{36}, h_{42}, h_{43}, h_{45}\}), \\ (e_5, \{h_2, h_4, h_8, h_9, h_{12}, h_{13}, h_{14}, h_{16}, h_{17}, h_{21}, h_{23}, h_{28}, h_{36}, h_{42}, h_{44}\}) \end{matrix} \right\}$$

and-product of (F, A) and (G, B)

$$(F, A) \wedge (G, B) = \left\{ \begin{matrix} \big((e_1, e_1), \{h_4, h_{21}\}\big), \big((e_1, e_2), \{h_4, h_7, h_{13}, h_{21}, h_{32}, h_{36}, h_{43}\}\big), \\ \big((e_1, e_5), \{h_4, h_{13}, h_{21}, h_{28}, h_{36}, h_{44}\}\big), \big((e_2, e_1), \{h_3, h_{21}, h_{22}, \\ h_{42}, h_{46}\}\big), \big((e_2, e_2), \{h_1, h_{13}, h_{21}, h_{32}, h_{36}, h_{42}\}\big), \big((e_2, e_5), \{h_{13}, \\ h_{21}, h_{28}, h_{36}, h_{42}, h_{44}\}\big), \big((e_4, e_1), \{h_3, h_{42}\}\big), \big((e_4, e_2), \{h_{13}, h_{15}, \\ h_{30}, h_{36}, h_{42}, h_{43}\}\big), \big((e_4, e_5), \{h_2, h_{13}, h_{23}, h_{28}, h_{36}, h_{42}\}\big), \\ \big((e_7, e_1), \{h_5, h_{34}\}\big), \big((e_7, e_2), \{h_1, h_{13}, h_{17}, h_{29}, h_{36}, h_{45}\}\big) \\ \big((e_7, e_5), \{h_{12}, h_{13}, h_{17}, h_{28}, h_{36}\}\big) \end{matrix} \right\}.$$

Subsequently, the Local and Global intersections are computed.

a) Since $A = \{e_1, e_2, e_4, e_7\}$, and $B = \{e_1, e_2, e_5\}$ then $A \cap B = \{e_1, e_2\}$. For each parameter x and $y \in A \cap B$, calculate $F(x) \cap G(y)$:

$$F(e_1) \cap G(e_1) = \{h_4, h_{21}\}$$

$$F(e_1) \cap G(e_2) = \{h_4, h_7, h_{13}, h_{21}, h_{32}, h_{36}, h_{43}\}$$

$$F(e_2) \cap G(e_1) = \{h_3, h_{21}, h_{22}, h_{42}, h_{46}\}$$

$$F(e_2) \cap G(e_2) = \{h_1, h_{13}, h_{21}, h_{32}, h_{36}, h_{42}\}$$

Then

$$\text{LUBI}_{int} = \bigcup_{x,y \in A\cap B} \big(F(x) \cap G(y)\big)$$

$$= \{h_1, h_3, h_4, h_7, h_{13}, h_{21}, h_{22}, h_{32}, h_{36}, h_{42}, h_{43}, h_{46}\}$$

b) For each $x \in A, y \in B$ such that $x, y \notin A \cap B$, calculate $F(x) \cap G(y)$:

$$F(e_4) \cap G(e_5) = \{h_2, h_{13}, h_{23}, h_{28}, h_{36}, h_{42}\}$$

$$F(e_7) \cap G(e_5) = \{h_{12}, h_{13}, h_{17}, h_{28}, h_{36}\}.$$

Then

$$\text{GEI}_{int} = \bigcup_{\substack{x\in A, y\in B\\ x,y\notin A\cap B}} \big(F(x) \cap G(y)\big) = \{h_2, h_{12}, h_{13}, h_{17}, h_{23}, h_{28}, h_{36}, h_{42}\}.$$

Finally, by combining the previous results, the Dual Consensus Decision (DCD-DM) is computed:

$$\text{DCD}_{int} = \bigcup_{x,y \in A\cap B} \big(F(x) \cap G(y)\big) \cap \bigcup_{\substack{x\in A, y\in B\\ x,y\notin A\cap B}} \big(F(x) \cap G(y)\big)$$

$$=\{h_1, h_3, h_4, h_7, h_{13}, h_{21}, h_{22}, h_{32}, h_{36}, h_{42}, h_{43}, h_{46}\} \cap \{h_2, h_{12}, h_{13}, h_{17}, h_{23}, h_{28}, h_{36}, h_{42}\}$$

$$= \{h_{13}, h_{36}, h_{42}\}.$$

3.3 Comparative Results and Implications for Decision-Making

3.3.1 Compared Decision Sets

Applying the Dual Consensus Decision method in the example provided leads to the following set of decisions

$$\text{DCD}_{int} = \{h_{13}, h_{36}, h_{42}\}.$$

In contrast, N. Çağman et al. [6] obtains a wider set of decisions when applying uni–int decision making method.

uni–int decision making = $\{h_4, h_{13}, h_{21}, h_{36}, h_{42}\}$.

3.3.2 Key Observations

a) **Precision vs. Breadth:**

i. DCD_{int} method: Yields higher precision by integrating only the candidates that appear both in intersecting and non-overlapping parameter pairs. The use of both shared (LUBI_{int}) and external (GEI_{int}) consensus intersections increase the decision confidence.

ii. uni–int method: Reflects a more inclusive approach by prioritizing union over intersection.

b) **Decision-Making Implications:**

i. Scores smaller than DCD_{int} method indicate higher confidence in the selected candidates due to the prioritization of shared and distinct evaluators' perspectives.

ii. uni–int method provides more alternatives that may be preferable when flexibility or expansion is required.

4. Some Property

Property 1: Symmetry of the Dual-Consensus Operator

Let $F_A = (F, A)$ and $F_B = (G, B)$ be two soft sets over the universe U. The Dual-Consensus Decision-DCD_{int} operator, which integrates both the Local Union of Binary Intersections-LUBI_{int} and the Global External Intersection-GEI_{int}, is symmetric with respect to its input soft sets. That is:

$$\mathrm{DCD}_{int}\ (F_A \wedge F_B) = \mathrm{DCD}_{int}\ (F_B \wedge F_A).$$

Proof:

The proof follows directly from the commutativity of set intersection, since both LUBI_{int} and GEI_{int}, are defined based on intersection operations. Therefore, swapping the input soft sets F_A and F_B does not affect the resulting decision set.

Property 2: De Morgan's Laws to DCD_{int}

Let $F_A = (F, A)$ and $F_B = (G, B)$ be two soft sets over the universe U. The complement of the Dual-Consensus Decision DCD_{int}^{c} is given by:

$$\mathrm{DCD}_{int}^{c} = \mathrm{LUBI}_{int}^{c} \cup \mathrm{GEI}_{int}^{c}.$$

Here, the superscript c denotes the classical complement in set theory.

Proof:

Since

$$\mathrm{LUBI}_{int} = \bigcup_{x,y \in A \cap B} \big(F(x) \cap G(y)\big)$$

Then

$$\text{LUBI}_{int}^{c} = \left(\bigcup_{x,y \in A \cap B} \big(F(x) \cap G(y)\big) \right)^{c} = \bigcap_{x,y \in A \cap B} \big(F(x) \cap G(y)\big)^{c}$$

$$= \bigcap_{x,y \in A \cap B} F(x)^{c} \cup G(y)^{c}.$$

And

$$\text{GEI}_{int} = \bigcup_{\substack{x \in A, y \in B \\ x,y \notin A \cap B}} \big(F(x) \cap G(y)\big)$$

Then

$$\text{GEI}_{int}^{c} = \left(\bigcup_{\substack{x \in A, y \in B \\ x,y \notin A \cap B}} \big(F(x) \cap G(y)\big) \right)^{c} = \bigcap_{\substack{x \in A, y \in B \\ x,y \notin A \cap B}} \big(F(x) \cap G(y)\big)^{c} = \bigcap_{\substack{x \in A, y \in B \\ x,y \notin A \cap B}} F(x)^{c} \cup G(y)^{c}.$$

Now

$$\text{DCD}_{int} = \bigcup_{x,y \in A \cap B} \big(F(x) \cap G(y)\big) \cap \bigcup_{\substack{x \in A, y \in B \\ x,y \notin A \cap B}} \big(F(x) \cap G(y)\big)$$

The complement of DCD_{int} defined By

$$\text{DCD}_{int}^{c} = \left(\bigcup_{x,y \in A \cap B} \big(F(x) \cap G(y)\big) \cap \bigcup_{\substack{x \in A, y \in B \\ x,y \notin A \cap B}} \big(F(x) \cap G(y)\big) \right)^{\boldsymbol{c}}$$

Applying De Morgan's laws to the definition of DCD_{int}^{c}, results in:

$$\text{DCD}_{int}^{c} = \left(\bigcup_{x,y \in A \cap B} \big(F(x) \cap G(y)\big) \right)^{c} \cup \left(\bigcup_{\substack{x \in A, y \in B \\ x,y \notin A \cap B}} \big(F(x) \cap G(y)\big) \right)^{c}$$

$$= \bigcap_{x,y \in A \cap B} \big(F(x) \cap G(y)\big)^{c} \cup \bigcap_{\substack{x \in A, y \in B \\ x,y \notin A \cap B}} \big(F(x) \cap G(y)\big)^{c}$$

$$= \bigcap_{x,y \in A \cap B} F(x)^{c} \cup G(y)^{c} \cup \bigcap_{\substack{x \in A, y \in B \\ x,y \notin A \cap B}} F(x)^{c} \cup G(y)^{c}$$

$$= \text{LUBI}_{int}^{c} \cup \text{GEI}_{int}^{c}.$$

5. Special Cases of Dual-Consensus

5.1 Generalization using union DCD_{uni}: In certain cases, expanding the set of viable alternatives may be beneficial. This can be accomplished by substituting the intersection operation with a union, thereby generating more inclusive decision outcomes. Such a modification increases the model's adaptability to diverse and complex scenarios. When applied to the previous example, using the union in place of the intersection results in an expanded decision set.

$$\mathrm{DCD}_{uni} = \bigcup_{x,y \in A \cap B} \big(F(x) \cap G(y)\big) \cup \bigcup_{\substack{x \in A, y \in B \\ x,y \notin A \cap B}} \big(F(x) \cap G(y)\big)$$

$$= \{h_1, h_3, h_4, h_7, h_{13}, h_{21}, h_{22}, h_{32}, h_{36}, h_{42}, h_{43}, h_{46}\} \cup \{h_2, h_{12}, h_{13}, h_{17,} h_{23}, h_{28}, h_{36}, h_{42}\}$$

$$= \{h_1, h_2, h_3, h_4, h_7, h_{12}, h_{13}, h_{17,} h_{21}, h_{22}, h_{23}, h_{28}, h_{32}, h_{36}, h_{42}, h_{43}, h_{46}\}.$$

Thus, obtain an expanded output is obtained compared to the output obtained using intersection alone, reflecting the model's flexibility in including more alternatives in the decision-making process.

Remark 1: it can be noted that

$$\mathrm{DCD}_{int} \subseteq \mathrm{DCD}_{uni}.$$

This results from the fact that the intersection of two sets is a subset of their union.

5.2 Handling the Case When $\mathrm{DCD}_{int} = \emptyset$

When applying the Dual Consensus Decision (DCD_{int}) method, it's possible that:

- $LUBI_{int}$ is empty, or
- GEI_{int} is empty, or
- Both are empty.

In such cases, the DCD_{int} result becomes empty, which may not be practical in decision-making scenarios. Although it is mathematically valid. This indicates no candidates satisfy consensus under both shared and distinct parameters, possibly due to evaluator disagreement, strict criteria, or poorly aligned parameter sets. To handle this, one may:

1- **Use fallback methods** like the more inclusive DCD_{uni} or uni–int.

2- **Reassess parameters** A and B to improve overlap or expand evaluator input.

3- **Partial DCD_{int}** if only one of $LUBI_{int}$ or GEI_{int} is empty: Consider using the non- empty one as a weakened decision, e.g.:

$$DCD_{int} = LUBI_{int} \text{ or } GEI_{int} \quad \text{(fallback mode)}$$

4- **Consensus Decision Method ($W - DCD_{int}$)**

The weighted Dual-Consensus Decision method with intersection-based logic ($W-DCD_{int}$) enables partial agreement between the Local Union of Binary Intersections ($LUBI_{int}$) and the Global External Intersection (GEI_{int}). Instead of requiring full consensus, this method assigns scores to candidates based on their presence in these intersected sets, allowing for a more flexible and inclusive decision-making process.

Steps of the $W-DCD_{int}$ Algorithm: Let (F,A) and (G,B) be two soft sets over universe U:

Step 1: Compute $LUBI_{int}$ and GEI_{int} int as in the original DCD_{int} method.

Step 2: Create a combined list of candidates appearing in either $LUBI_{int}$ or GEI_{int}.

Step 3: Assign a score to each candidate $h_i \in U$ as follows:

$$Score(h_i) = \begin{cases} 2 & if h_i \in LUBI_{int} \cap GEI_{int} \\ 1 & if h_i \in LUBI_{int} \cup GEI_{int}, h_i \notin both \\ 0 & if h_i \notin LUBI_{int} \; and \; h_i \notin GEI_{int} \end{cases}.$$

Step 4: Compute weight w_{h_i} of element h_i such that the weight w_{h_i} defined by the ratio between of the number of parameter pairs (x,y) where h_i appears in $F(x) \cap G(y)$ to the total number of pairs (x,y) for all x and $y \in A \cup B$. Mathematically:

$$w_{h_i} = \frac{|\{(x,y) \in \mathrm{A} \times \mathrm{B}: h_i \in F(x) \cap G(y)\}|}{|\{(x,y) \in \mathrm{A} \times \mathrm{B}\}|}.$$

Where $|\cdot|$ denotes set cardinality (number of elements).

Step 5: Compute $w_{h_i}.Score(h_i)$ for all h_i.

Step 6: Set a threshold $\theta \in [0,2]$ to select candidates. (θ may be determined by domain experts, or set according to the specific objectives and criteria of stakeholders (e.g., companies or decision-makers)).

Step 7: The final decision set ($W-DCD_{int}$) consists of all candidates with scores equal to or above the threshold θ.

Referring back to Example 1, there is

$$LUBI_{int} = \{h_1, h_3, h_4, h_7, h_{13}, h_{21}, h_{22}, h_{32}, h_{36}, h_{42}, h_{43}, h_{46}\}$$

$$GEI_{int} = \{h_2, h_{12}, h_{13}, h_{17}, h_{23}, h_{28}, h_{36}, h_{42}\}$$

Subsequently, score each candidate is

Candidate	**$\mathbf{LUBI}_{int}$**	**GEI_{int}**	Score
h_{13}	Yes	yes	2
h_{36}	Yes	yes	2

Candidate	$\mathbf{LUBI}_{int}$	GEI_{int}	Score
h_{42}	Yes	yes	2
$h_1, h_3, h_4, h_7, h_{21}, h_{22}, h_{32}, h_{43}, h_{46}$	Yes	No	1
$h_2, h_{12}, h_{17}, h_{23}, h_{28}$	No	yes	1

Next, the weight w_{h_i} of each element h_i is computed. For example, if $i = 1$,

- There are 2 pairs where h_1 appears in the intersection i.e.,
  $$|\{(x, y) \in A \times B: h_1 \in F(x) \cap G(y)\}| = 2.$$
- There are 12 pairs with intersections i.e., $|\{(x, y) \in A \times B\}| = 12$.

  Then
  $$w_{h_1} = \frac{2}{12} = 0.16.$$

Similarly, for any **i** the weight w_{h_i} of h_i is determined using the same method. Accordingly,

$w_{h_2} = \frac{1}{12} = 0.08, w_{h_3} = \frac{2}{12} = 0.16, w_{h_4} = \frac{3}{12} = 0.25, w_{h_7} = \frac{1}{12} = 0.08, w_{h_{12}} = \frac{1}{12} = 0.08, w_{h_{13}} = \frac{8}{12} = 0.66, w_{h_{17}} = \frac{2}{12} = 0.16, w_{h_{21}} = \frac{6}{12} = 0.50, w_{h_{22}} = \frac{1}{12} = 0.08, w_{h_{23}} = \frac{1}{12} = 0.08, w_{h_{28}} = \frac{4}{12} = 0.33, w_{h_{32}} = \frac{2}{12} = 0.16, w_{h_{36}} = \frac{8}{12} = 0.66, w_{h_{42}} = \frac{6}{12} = 0.50, w_{h_{43}} = \frac{2}{12} = 0.16, w_{h_{46}} = \frac{1}{12} = 0.08.$

Then, $w_{h_i}.Score(h_i)$ is computed for all h_i.

Candidate	$\boldsymbol{w_{h_i}}$	$\boldsymbol{Score(h_i)}$	$\boldsymbol{w_{h_i}.Score(h_i)}$
h_{13}, h_{36}	0.66	2	$0.66 \times 2 = 1.32$
h_{42}	0.50	2	$0.50 \times 2 = 1$
h_{21}	0.50	1	$0.50 \times 1 = 0.50$
h_{28}	0.33	1	$0.33 \times 1 = 0.33$
h_4	0.25	1	$0.25 \times 1 = 0.25$
$h_1, h_2, h_3, h_{17}, h_{32}, h_{43}$	0.16	1	$0.16 \times 1 = 0.16$
$h_7, h_{12}, h_{22}, h_{23}, h_{46}$	0.08	1	$0.08 \times 1 = 0.08$

Next, set $\theta = 0.5$ as the threshold. Consequently, the decision set consists of all elements h_i such that $w_{h_i}.Score(h_i) \geq 0.5$. Therefore,

$$W - DCD_{int} = \{ h_{13}, h_{21}, h_{36}, h_{42}\}$$

This method prevents the decision from being empty and provides graded, confidence-aware outputs. Introduce weights or thresholds to soften the strict intersection rule in the DCD method. Instead of requiring elements to exist in all relevant intersections (crisp logic), allow partial membership or agreement turning the model from crisp to fuzzy or graded. This makes the method more flexible and tolerant to minor disagreements, especially useful when strict consensus leads to an empty decision set.

6. Introducing Weights and Fuzzy Relaxation to DCD-DM: Toward a Fuzzy-Soft Decision

The classical DCD-DM framework requires strict agreement across decision makers, meaning alternatives must meet both local and global consensus conditions. This strictness can lead to no decisions when evaluations conflict or are incomplete. To improve this, fuzzy logic is integrated into the soft set structure, allowing partial agreement through weights and confidence levels. This changes rigid yes/no decisions into flexible confidence scores, creating a more adaptable and robust decision-making process under uncertainty. This soft-fuzzy extension preserves the foundational logic of DCD-DM while enabling nuanced decision aggregation, making it more suitable for complex, real-world decision-making environments where ambiguity and partial consensus are common.

Remark 2: A soft fuzzy set is a mathematical structure that integrates the concepts of fuzzy sets and soft sets by defining a fuzzy relation between a universal set U and a parameter set E. Detailed discussions can be found in [13].

6.1 Soft Fuzzy Sets and Relaxed DCD

Let (F, A) and (G, B) be two soft fuzzy sets over universe U with parameter sets A and $B \subseteq E$. For each mapping $F: A \rightarrow P(U), G: B \rightarrow P(U),$ The associated fuzzy relation is defined as a mapping:

$$\mu_F: A \times U \rightarrow [0,1], \quad \mu_G: B \times U \rightarrow [0,1].$$

These represent degrees of satisfaction for each element $h_i \in U$ with respect to each parameter $e_j \in A \cup B$. Instead of computing crisp intersections like:

$$\mathrm{DCD}_{int} = \bigcup_{x,y \in A \cap B} \big(F(x) \cap G(y)\big) \cap \bigcup_{\substack{x \in A, y \in B \\ x,y \notin A \cap B}} \big(F(x) \cap G(y)\big)$$

A fuzzy consensus degree $\delta(h_i) \in [0,1]$ is defined for each object $h_i \in U$, and computed as:

$$\delta(h_i) = \mu_{\mathbf{DCD}_{int}}(h_i) = w_1 \cdot \mu_{LUBI}(h_i) + w_2 \cdot \mu_{GEI}(h_i), w_1 + w_2 = 1.$$

This allows ranking of objects based on their fuzzy consensus scores.

Where:

a) $w_1, w_2 \in [0,1]$: Are weights reflecting the importance of local (LUBI_{int}) and global (GEI_{int}) consensus.

b) $\mu_{LUBI}(h_i)$: Fuzzy degree of membership in LUBI_{int} i.e.,

$$\mu_{LUBI}(h_i) = min\left(\mu_F(e_j, h_i),\ \mu_G(e_j, h_i)\right), e_j \in A \cap B, h_i \in U$$

c) $\mu_{GEI}(h_i)$: Fuzzy degree of membership in GEI_{int} i.e.,

$$\mu_{GEI}(h_i) = min(\mu_F(e_m, h_i),\ \mu_G(e_n, h_i)), e_m \in A, e_n \in B\ s.t\ e_m, e_n \notin A \cap B, h_i \in U.$$

Example 2: Soft-Fuzzy DCD-DM for Investment Selection

Setup: Let the universe of alternatives be:

$$U = \{h_1, h_2, h_3, h_4\} \text{ (Investment projects)}$$

Let the evaluation parameters be:

$$E = \{e_1\text{: high return}, e_2\text{: low risk}, e_3\text{: high liquidity}\}.$$

Two analysts evaluate the alternatives:

Analyst 1 (Soft fuzzy Set (F, A) with $A = \{e_1, e_2\}$):

$$\mu_F(e_1, h_1) = 0.9, \mu_F(e_1, h_4) = 0.8, \mu_F(e_2, h_3) = 0.7, \mu_F(e_2, h_4) = 0.9.$$

Analyst 2 (Soft fuzzy Set (G, B) with $B = \{e_2, e_3\}$):

$$\mu_G(e_2, h_1) = 0.6, \mu_G(e_3, h_2) = 0.8, \mu_G(e_3, h_3) = 0.6$$

First, compute fuzzy LUBI_{int} and GEI_{int} scores

LUBI (Local Union of Binary Intersections): only $e_2 \in A \cap B$, for each $h_i \in U$, define:

$$\mu_{LUBI}(h_i) = min(\mu_F(e_2, h_i),\ \mu_G(e_2, h_i))$$

Results:

$$\mu_{LUBI}(h_1) = min(0,0.6) = 0, \mu_{LUBI}(h_2) = min(0,0) = 0,$$

$$\mu_{LUBI}(h_3) = min(0.7,0) = 0, \mu_{LUBI}(h_4) = min(0.9,0) = 0.$$

GEI (Global External Intersection): for mismatched parameters: (e_1, e_3)

$$\mu_{GEI}(h_i) = min(\mu_F(e_1, h_i),\ \mu_G(e_3, h_i))$$

Results:

$$\mu_{GEI}(h_1) = min(0.9,0) = 0,\ \mu_{GEI}(h_2) = min(0,0.8) = 0$$

$$\mu_{GEI}(h_3) = min(0,0.6) = 0, \mu_{GEI}(h_4) = min(0.8,0) = 0$$

Both LUBI and GEI return zero values, indicating the absence of valid intersections. In such cases, fuzzy relaxation is applied by replacing intersections with fuzzy unions combined with appropriate weighting. Fuzzy unions are computed in place of intersections to ensure continuity in decision-making. For each $h_i \in U$:

$$\mu_{LUBI}(h_i) = max\big(\mu_F(e_2, h_i),\ \mu_G(e_2, h_i)\big)$$
$$\mu_{GEI}(h_i) = max\big(\mu_F(e_1, h_i),\ \mu_G(e_3, h_i)\big)$$

LUBI (Fuzzy union for e_2):

$$\mu_{LUBI}(h_1) = max(0{,}0.6) = 0.6, \mu_{LUBI}(h_2) = max(0{,}0) = 0$$

$$\mu_{LUBI}(h_3) = max(0.7, 0) = 0.7, \mu_{LUBI}(h_4) = max(0.9{,}0) = 0.9$$

GEI (Fuzzy union for e_1, e_3):

$$\mu_{GEI}(h_1) = max(0.9, 0) = 0.9, \mu_{GEI}(h_2) = max(0{,}0.8) = 0.8$$

$$\mu_{GEI}(h_3) = max(0, 0.6) = 0.6,\ \mu_{GEI}(h_4) = max(0.8{,}0) = 0.8$$

Next, compute final fuzzy score ($w_1 = w_2 = 0.5$)

$$\delta(h_i) = 0.5 \cdot \mu_{LUBI}(h_i) + 0.5 \cdot \mu_{GEI}(h_i)$$

h_i	$\mu_{LUBI}(h_i)$	$\mu_{GEI}(h_i)$	$\delta(h_i)$
h_1	0.6	0.9	0.75
h_2	0	0.8	0.4
h_3	0.7	0.6	0.65
h_4	0.9	0.8	0.85

Then, Ranking Result: The top-ranked alternative is h_4, followed by h_1, h_3, and h_2, making h_4 the most recommended choice.

The proposed fuzzy-soft extension of the DCD-DM model offers several practical advantages. By avoiding degenerate outcomes such as empty decision sets, it ensures that meaningful recommendations can always be generated. Furthermore, the introduction of fuzzy consensus scores enables nuanced ranking of alternatives, providing decision-makers with confidence levels rather than binary outputs. This capability is especially valuable in environments characterized by ambiguity and partial information. However, the model's effectiveness relies on the accurate calibration of fuzzy membership values, which may introduce subjectivity or variability. Additionally, selecting appropriate weights w_1 and w_2 for combining local and global consensus remains a subjective task that could influence final outcomes. Finally, the fuzzy extension introduces increased computational complexity compared to its crisp counterpart, which may limit scalability in large-scale decision systems.

7. Discussion and Conclusion

The DCD-DM model presents a mathematically grounded and operationally flexible approach to decision making, integrating dual consensus strategies to harness both localized and global agreement structures. By embedding soft-fuzzy set theory within the DCD-DM framework, the methodology achieves enhanced expressiveness, permitting fine-grained control over uncertainty and confidence levels in multi-criteria environments. This fusion of binary intersection logic with soft-fuzzy reasoning demonstrates superior adaptability in complex decision spaces. Future research may explore its extension to dynamic or real-time decision systems, optimization scenarios, and broader AI-based inference engines.

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