Hankel determinant problem for a new subclass of analytic functions defined by integral operator associated with the Hurwitz-Lerch zeta function

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Submitted: 09/10/2024. Accepted: 23/11/2024. Published: 01/12/2024

Abstract

In this paper, we introduce a new subclass of analytic functions defined by a generalized integral operator associated with the Hurwitz-Lerch zeta function. Sharp bound for the functional belong to the subclass are found. Our result extends the corresponding previously known results

Keywords: Hankel determinant; univalent functions; integral operator; Hurwitz-Lerch Zeta function.

1. Introduction

The Hankel determinant is a specific type of determinant associated with Hankel matrices. It is useful in various fields of mathematics and applied sciences and has also been considered by several authors. Janteng, et al (Jantengh.et.at, 2006) ave considered the functional $|a_2a_4 - a_3^2|$ and found a sharp bound for the function f in the subclass $\mathfrak{R}$ of S in the unit disk. In their work, they have shown $|a_2a_4 - a_3^2| < \frac{4}{9}$ for $f \in \mathfrak{R}$. (Jantengh.et.at, 2007) obtained the second Hankel determinant and sharp bounds for the subclasses namely, starlike and convex functions denoted by ST and CV and have shown that $|a_2a_4 - a_3^2| < 1$ and $|a_2a_4 - a_3^2| < \frac{1}{8}$ respectively. Following this study, other scholars have focused on deriving sharp upper bounds for $H_2(2)$, to name (Yavuz,2015) and (Jae Ho Choi,2020)) who investigated analytic functions defined by the Ruschewey derivative and established an upper bound for the second Hankel determinant in different classes. To add more, (Mohammed& Darus, 2012) have used the linear operator for a new class of analytic functions to determine an upper bound. Also (Kund &Mishra, 2013) examined a class of analytic functions associated with the Carlson-Shaffer operator in the unit disk to estimates the second Hankel determinant. Furthermore, (Ehrenborg,2000) studied the Hankel determinant of exponential polynomials. (Alkabaily& Alabbar, 2020) introduced class of analytic functions defined by generalized Srivastava–Attiya operator to obtain an upper bound to the second Hankel Determinant. (Amer ,2016) Having the linear operator, there are interesting properties of normalized function in the unit disk for sharp second Hankel for linear operator.

In this paper, we upper bounds of the second Hankel determinant $|a_2a_4 - a_3^2|$ for functions belonging to the new subclass denoted by $\mathcal{R}_{s,b}^{\alpha}(\rho,\beta)$.

The Hankel determinant of f for $q \geq 1$ and $k \geq 1$ was defined by (Noonan &Thomas, 1976) as

$$H_q(k) = \begin{vmatrix} a_k & a_{k+1} & \cdots & a_{k+q-1} \\ a_{k+1} & a_{k+2} & \cdots & a_{k+q-2} \\ \vdots & \vdots & \cdots & \vdots \\ a_{k+q-1} & a_{k+q} & \cdots & a_{k+2q-2} \end{vmatrix}.$$

The determinant $H_q(k)$ has been extensively studied, with $H_2(2)$ referring to the second Hankel determinant which is defined by $|a_2a_4 - a_3^2|$.

Let A denote the class of all analytic functions in the open unit disk

$$\mathrm{U} = \{z \in \mathbb{C}: |z| < 1\},$$

and given by the normalized power series

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k. \qquad (1.1)$$

Let S denote the subclasses of A consisting of univalent functions.

For function $f \in \mathrm{A}$ given by (1.1) and $g \in \mathrm{A}$ given by $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, we define the Hadamard product (or convolution) of f and g given by the power series

$$(f * g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$

1.1. Definition (Srivastava and Choi, 2001)

A general Hurwitz–Lerch Zeta function $\Phi(z,s,b)$ defined by

$$\Phi(z,s,b) = \sum_{k=0}^{\infty} \frac{z^k}{(k+b)^s},$$

where $(s \in \mathbb{C}, b \in \mathbb{C} - \mathbb{Z}_0^-)$ when $(|z| < 1)$, and $(\Re(s) > 1)$ when $(|z| = 1)$.

(Nagat & Darus,2011,2012) introduced a general integral operator $\mathfrak{J}_{s,b}^{\alpha} f(z)$ which is defined by means of a general Hurwitz Lerch Zeta. As we will show in the following:

For $(s \in \mathbb{C}, b \in \mathbb{C} - \mathbb{Z}_0^-)$ the generalized integral operator $\mathfrak{J}_{s,b}^{\alpha} f(z)$: $\mathrm{A} \to \mathrm{A}$ is defined by

$$\Im_{s,b}^{\alpha} f(z) = \Gamma(2-\alpha) z^{\alpha} D_z^{\alpha} \Phi^{*}(z,s,b), \quad (\alpha \neq 2,3,4,\cdots)$$

$$= z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} \left(\frac{b}{k-1+b} \right)^{s} a_k z^k, \quad (z \in \mathrm{U}). \qquad (1.2)$$

Special cases of this operator includes:

1) $\Im_{0,b}^{\alpha} f(z) \equiv \Omega^{\alpha} f(z) = z + \sum_{k=2}^{\infty} \frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)} a_k z^k$,

is Owa and Srivastava operator (Owa & Srivastava ,1984).

2) $\Im_{s,b+1}^{0} f(z) \equiv J_{s,b} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{b+1}{k+b} \right)^{s} a_k z^k$,

is Srivastava and Attiya integral operator (Srivastava & Attiya, 2007).

3) $\Im_{1,1}^{0} f(z) \equiv A(f)(z) = \int_0^z \frac{f(t)}{t} dt = z + \sum_{k=2}^{\infty} \frac{1}{k} a_k z^k$,

is Alexander integral operators (Alexander,1915).

4) $\Im_{1,2}^{0} f(z) \equiv L(f)(z) = \frac{2}{z} \int_0^z f(t) dt = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1} \right) a_k z^k$,

is Libera integral operators (Libera ,1969).

5) $\Im_{\sigma,2}^{0} f(z) \equiv I^{\sigma} f(z) = z + \sum_{k=2}^{\infty} \left(\frac{2}{k+1} \right)^{\sigma} a_k z^k$,

is Jung– Kim– Srivastava integral operator (Jung .et.at,1993).

By using our integral operator, we introduce the following subclass of A.

A function $f \in \mathrm{A}$ is said to be in the subclass $\mathcal{R}_{s,b}^{\alpha}(\rho, \beta)$

For $s \in \mathbb{C}, b \in \mathbb{C} - z_0^{-}$, and $(\alpha \neq 2,3,4, \dots)$, $0 \leq \rho \leq 1$ and $\frac{-\pi}{2} \leq \beta \leq \frac{\pi}{2}$,

if and only if

$$\Re \left\{ e^{i\beta} \frac{(\Im_{s,b}^{\alpha} f(z))}{z} \right\} > \rho \cos \beta, \quad (z \in \mathrm{U}). \qquad (1.3)$$

By suitably specializing the values of α, b and s in the subclass $\mathcal{R}_{s,b}^{\alpha}(\rho, \beta)$

We note that,

$$\mathcal{R}_{0,b}^{0}(\rho, \beta) = \left\{ f : f \in \mathrm{A} \text{ and } \Re \left\{ e^{i\beta} \frac{f(z)}{z} \right\} > \rho\cos\beta, \quad z \in \mathrm{U} \right\}.$$

Let P be the family of functions $p \in P$ satisfying $p(o) = 1$ and $\Re\{p(o)\} > 0$, and

$$p(z) = 1 + c_1 z + c_2 z^2 + \ldots, \quad z \in U. \quad (1.4$$

It follows from (1.3) that

$$f \in \Re_{s,b}^{\alpha}(\rho,\beta) \Leftrightarrow e^{i\beta} \frac{\Im_{s,b}^{\alpha} f(z)}{z} = [(1-\rho) + p(z)] + \rho(cos\beta + i\ sin\beta).$$

In the present paper, we consider the second Hankel determinant for functions f belong to the subclass $\mathcal{R}_{s,b}^{\alpha}(\rho,\beta)$.

2. Materials and Methods

To establish our results, we recall the following:

2.1. Lemma (Duren,1983)

If $p \in P$ then $|c_k| \leq 2$, for all $k \in \mathbb{N}$.

2.2. Lemma (Libera & E. J. Zlotkiewicz,1983)

Let the function $p \in P$ be given by (1.4). Then

$$2c_2 = c_1^2 + x(4 - c_2),$$

for some x , $|x| \leq 1$, and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1 x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)z,$$

for some z , $|z| \leq 1$.

3. Results and Discussion

3.1. Theorem

Let the function f given by (1.1) and belongs to the class $\mathcal{R}_{s,b}^{\alpha}(\rho,\beta)$

For $s \in \mathbb{C}, b \in \mathbb{C} - z_0^-$, and $(\alpha \neq 2,3,4, \ldots)$, $0 \leq \rho \leq 1$ and $\frac{-\pi}{2} \leq \beta \leq \frac{\pi}{2}$,

then

$$|a_2 a_4 - a_3^2| \leq (1-\rho)^2 \cos^2\beta \left[\frac{(2-\alpha)^2 (3-\alpha)^2 (b+2)^{2s}}{9} \right]$$

Proof

Let f given by (1.1) and belongs to the class $\mathcal{R}_{s,b}^{\alpha}(\rho,\beta)$.

Then,

$$f \in \mathcal{R}_{s,b}^{\alpha}(\rho,\beta), \Leftrightarrow e^{i\beta} \frac{\Im_{s,b}^{\alpha} f(z)}{z} = [(1-\rho)p(z) + \rho](cos\,\beta + i\ sin\,\beta),$$

where $p \in P$ and is given by Lemma 1 , we rewrite as

$$e^{i\beta}\left(1+\sum_{k=2}^{\infty}\frac{\Gamma(k+1)\Gamma(2-\alpha)}{\Gamma(k+1-\alpha)}a_k z^{k-1}\right) = [(1-\rho)+\rho\left(1+\sum_{k=2}^{\infty}c_k z^k\right)](\cos\beta + i\sin\beta)$$

Comparing the coefficients, we get

$$\left.\begin{aligned}
e^{i\beta}\frac{\Gamma(3)\Gamma(2-\alpha)}{\Gamma(3-\alpha)}\left(\frac{b}{b+1}\right)^s a_2 &= (1-\rho)c_1\cos\beta,\\
e^{i\beta}\frac{\Gamma(4)\Gamma(2-\alpha)}{\Gamma(4-\alpha)}\left(\frac{b}{b+2}\right)^s a_3 &= (1-\rho)c_2\cos\beta,\\
e^{i\beta}\frac{\Gamma(5)\Gamma(2-\alpha)}{\Gamma(5-\alpha)}\left(\frac{b}{b+3}\right)^s a_4 &= (1-\rho)c_3\cos\beta.
\end{aligned}\right\}$$

$$a_2 = \frac{(1-\rho)c_1\cos\beta\ \Gamma(3-\alpha)(b+1)^s}{e^{i\beta}\Gamma(3)\Gamma(2-\alpha)b^s}$$

$$a_3 = \frac{(1-\rho)c_2\cos\beta\,\Gamma(4-\alpha)(2+b)^s}{e^{i\beta}\Gamma(4)\Gamma(2-\alpha)b^s}$$

$$a_4 = \frac{(1-\rho)c_3\cos\beta\,\Gamma(5-\alpha)(b+3)^s}{e^{i\beta}\Gamma(5)\Gamma(2-\alpha)b^s}$$

$$|a_2a_4 - a_3^2| = \left|\frac{(1-\rho)c_1\cos\beta\ \Gamma(3-\alpha)(b+1)^s}{e^{i\beta}\Gamma(3)\Gamma(2-\alpha)b^s}\ \frac{(1-\rho)c_3\cos\beta\,\Gamma(5-\alpha)(b+3)^s}{e^{i\beta}\Gamma(5)\Gamma(2-\alpha)b^s} - \frac{(1-\rho)^2c_2^2\cos^2\beta\left(\Gamma(4-\alpha)\right)^2(2+b)^{2s}}{e^{i2\beta}\left(\Gamma(4)\right)^2(\Gamma(2-\alpha))^2b^{2s}}\right|$$

$$= \left|\frac{(1-\rho)^2\cos^2\beta}{e^{i2\beta}(\Gamma(2-\alpha))^2b^{2s}}\right|\left|\frac{c_1c_3\Gamma(3-\alpha)(b+1)^s\Gamma(5-\alpha)(b+3)^s}{\Gamma(3)\Gamma(5)} - \frac{c_2^2\left(\Gamma(4-\alpha)\right)^2(2+b)^{2s}}{\left(\Gamma(4)\right)^2}\right|.$$

Now assuming $c_1 = c$, $(0 \le c \le 2)$ and using lemma 2.1 we have

$$|a_2a_4 - a_3^2| = \left|\frac{(1-\rho)^2\cos^2\beta}{e^{i2\beta}\left(\Gamma(2-\alpha)\right)^2b^{2s}}\right|$$

$$\left|\frac{c[c^3+2(4-c^2)cx - cx^2(4-c^2)+2(4-c^2)(1-|x|^2)z]\Gamma(3-\alpha)(b+1)^s\Gamma(5-\alpha)(b+3)^s}{4\,\Gamma(3)\Gamma(5)} - \frac{[c^4+2xc^2(4-c^2)+x^2(4-c^2)^2]\left(\Gamma(4-\alpha)\right)^2(2+b)^{2s}}{4\left(\Gamma(4)\right)^2}\right|$$

$$\left|a_2a_4-a_3^2\right| = \left|\frac{(1-\rho)^2\cos^2\beta}{e^{i2\beta}\left(\Gamma(2-\alpha)\right)^2 b^{2s}}\right| \frac{c^4\Gamma(3-\alpha)(b+1)^s\Gamma(5-\alpha)(b+3)^s}{4\Gamma(3)\Gamma(5)}$$

$$+\frac{\Gamma(3-\alpha)(b+1)^s\Gamma(5-\alpha)(b+3)^s}{4\Gamma(3)\Gamma(5)}2(4-c^2)c^2x$$

$$-\frac{\Gamma(3-\alpha)(b+1)^s\Gamma(5-\alpha)(b+3)^s}{4\Gamma(3)\Gamma(5)}c^2x^2(4-c^2)$$

$$+\frac{\Gamma(3-\alpha)(b+1)^s\Gamma(5-\alpha)(b+3)^s}{4\Gamma(3)\Gamma(5)}2c(4-c^2)(1-|x|^2)z-\frac{c^4\left(\Gamma(4-\alpha)\right)^2(2+b)^{2s}}{4\left(\Gamma(4)\right)^2}$$

$$-\frac{\left(\Gamma(4-\alpha)\right)^2(2+b)^{2s}}{4\left(\Gamma(4)\right)^2}2xc^2(4-c^2)-\frac{\left(\Gamma(4-\alpha)\right)^2(2+b)^{2s}}{4\left(\Gamma(4)\right)^2}\ x^2(4-c^2)^2$$

$$\left|a_2a_4-a_3^2\right| = \left|\frac{(1-\rho)^2\cos^2\beta}{e^{i2\beta}(\Gamma(2-\alpha))^2 b^{2s}}\right|\left|\left[\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s}{4\Gamma(3)\Gamma(5)}-\frac{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}}{4(\Gamma(4))^2}\right]c^4\right.$$

$$+\left[\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s}{4\Gamma(3)\Gamma(5)}-\frac{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}}{4\left(\Gamma(4)\right)^2}\right]xc^2(4-c^2)$$

$$-\left[\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s c^2}{4\Gamma(3)\Gamma(5)}+\frac{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}(4-c^2)}{4\left(\Gamma(4)\right)^2}\right]x^2(4-c^2)$$

$$\left.+\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s c(4-c^2)(1-|x|^2)z}{2\Gamma(3)\Gamma(5)}\right|.$$

An application of triangle inequality and replacement of $|x|$ by μ give

$$\left|a_2a_4-a_3^2\right| \le \frac{(1-\rho)^2\cos^2\beta}{e^{i2\beta}\left(\Gamma(2-\alpha)\right)^2 b^{2s}}\left[\left[\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s}{2\Gamma(3)\Gamma(5)}-\frac{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}}{2\left(\Gamma(4)\right)^2}\right]c^4\right.$$

$$+\left[\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s}{2\Gamma(3)\Gamma(5)}-\frac{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}}{2\left(\Gamma(4)\right)^2}\right]\mu c^2(4-c^2)$$

$$+\left[\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s c^2}{4\Gamma(3)\Gamma(5)}-\frac{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}(4-c^2)}{4\left(\Gamma(4)\right)^2}\right]\mu^2(4-c^2)$$

$$+\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s c(4-c^2)}{2\Gamma(3)\Gamma(5)}$$

$$\left.-\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s\mu^2 c(4-c^2)}{2\Gamma(3)\Gamma(5)}\right],$$

where $0 \le c \le 2$ and $0 \le \mu \le 1$.

We next maximize the function $F(c,\mu)$ on the closed square $[0,2]\times[0,1]$. Since

$$F'(\mu)=\frac{(1-\rho)^2\cos^2\beta}{e^{i2\beta}\left(\Gamma(2-\alpha)\right)^2 b^{2s}}\left(\left[\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s}{2\Gamma(3)\Gamma(5)}-\frac{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}}{2\left(\Gamma(4)\right)^2}\right]\mu c^2(4-c^2)\right.$$
$$\left.+\left[\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s}{2\Gamma(3)\Gamma(5)}+\frac{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}(4-c^2)}{2\left(\Gamma(4)\right)^2}\mu(4-c^2)\right]\right)$$

$$\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s\left(\Gamma(4)\right)^2-\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}\Gamma(3)\Gamma(5)\geq 0$$

$$\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s\left(\Gamma(4)\right)^2\geq\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}\Gamma(3)\Gamma(5)$$

$$\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s}{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}}\geq\frac{\Gamma(3)\Gamma(5)}{\left(\Gamma(4)\right)^2}$$

$$\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s}{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}}\geq\frac{4}{3}$$

$$\frac{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}}{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s}\leq\frac{3}{4}$$

Then we have $(\frac{\partial F}{\partial\mu}>0)$, for $(0\leq\mu\leq 1)$.

Thus $\frac{\partial F}{\partial\mu}$ cannot have a maximum in the interior of the closed square $[0,2]\times[0,1]$.

Moreover, for fixed $c\in[0,2]$, $(max_{0\leq\mu\leq 1}F(c,\mu)=F(1,\mu)=G(c)$

$$G(c)=\frac{(1-\rho)^2\cos^2\beta}{e^{i2\beta}\left(\Gamma(2-\alpha)\right)^2 b^{2s}}\left[\left[\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s}{4\Gamma(3)\Gamma(5)}-\frac{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}}{4\left(\Gamma(4)\right)^2}\right]c^4\right.$$
$$+\left[\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s}{2\Gamma(3)\Gamma(5)}-\frac{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}}{2\left(\Gamma(4)\right)^2}\right]c^2(4-c^2)$$
$$\left.+\left[\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s c^2}{4\Gamma(3)\Gamma(5)}+\frac{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}(4-c^2)}{4\left(\Gamma(4)\right)^2}\right](4-c^2)\right],$$

so

$$G'(c)=\frac{(1-\rho)^2\cos^2\beta}{e^{i2\beta}(\Gamma(2-\alpha))^2b^{2s}}\left[\left[\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s}{4\Gamma(3)\Gamma(5)}-\frac{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}}{4(\Gamma(4))^2}\right]4c^3\right.$$

$$+\left[\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s}{2\Gamma(3)\Gamma(5)}-\frac{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}}{2\left(\Gamma(4)\right)^2}\right][c^2(-2c)+(4-c^2)(2c)]$$

$$+\left[\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s}{4\Gamma(3)\Gamma(5)}[c^2(-2c)+(4-c^2)(2c)]\right.$$

$$\left.\left.+\frac{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}2(4-c^2)(-2c)}{4\left(\Gamma(4)\right)^2}\right]\right]$$

$$G'(c)=\frac{2(1-\rho)^2\cos^2\beta}{e^{i2\beta}\left(\Gamma(2-\alpha)\right)^2b^{2s}}\left[\left(\frac{\Gamma(5-\alpha)(b+3)^s\Gamma(3-\alpha)(b+1)^s}{\Gamma(3)\Gamma(5)}\right)c(3-c^2)-\left(\frac{\left(\Gamma(4-\alpha)\right)^2(b+2)^{2s}}{4\left(\Gamma(4)\right)^2}\right)c(4-c^2)\right],$$

so that $G'(c)$ for $0\le c\le 2$ and has real critical point at $c=0$.

Therefore, $max_{0\le c\le 2}\,G(c)$ occurs at $c=0$

Therefore, the upper bound of corresponds to $\mu=1$ and $c=0$.

Hence

$$\left|a_2a_4-a_3^2\right|\le(1-\rho)^2\cos^2\beta\left[\frac{(2-\alpha)^2(3-\alpha)^2(b+2)^{2s}}{9}\right].$$

The proof of Theorem 3.1 is complete.

Taking $s=0$ and $\alpha=0$ in Theorem 3.1 , we have the next result reduces to a result of (Janteng.et al ,2006**)**

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3.2. Corollary

Let the function f given by 1.1 and belongs to the class $\Re_\lambda(\rho)$,

Then

$$|a_2a_4-a_3^2|\le\left[\frac{(1-\rho)^2(2-\lambda)^2(3-\lambda)^2}{9}\right].$$

Taking $\rho=\beta=s=0$ and $\alpha=1$ in Theorem 3.1, we have the next result reduces to a result of (Mishra & Gochhayat, 2008)**.**

3.3. Corollary

Let the function f given by 1.1 and belongs to the class $\mathfrak{R}$,

Then $|a_2a_4 - a_3^2| \leq \frac{4}{9}$.

Conclusion:

We have introduced a new subclass of analytic functions defined by a generalized integral operator related to the Hurwitz-Lerch zeta function. We have also established exact bounds for the functionals within this subclass. These results advance our understanding of analytic functions and provide new tools for exploring their properties. Future research could build on these findings to uncover further insights and applications. Many other work on analytic functions related to operator can be see [(Najat.et.at,2023),(Amer.et.at,2024),(Alabber.et.at,2023),(Shmella& Amer,2024)].

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