# Estimation of the Bounds of Univalent Functional of Coefficients Apply the Subordination Method

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**ABSTRACT**:

In this paper ,we Introduce a class of analytic and bi-univalent functions in the open unit disk $\mathbb{U} = \{\boldsymbol{z}: |\boldsymbol{z}| < \mathbf{1}\,; \boldsymbol{z} \in \mathbb{C}\}$ by using a generalized derivative operator (Amer & Darus, 2011) and apply the subordination method to the functional of coefficients problem .

Furthermore, an estimation for the initial Taylor and Maclaurin coefficients of functions in $\boldsymbol{\mathcal{H}^{q}_{\Sigma}(\lambda, \alpha)}$ was given.

**Keywords:** Analytic functions; Univalent function ; Unit disk..

## INTRODUCTION:

Let $\mathcal{A}$ denote the class of functions $f$ in the open unit disk

$$\mathbb{U} = \{z: |z| < 1; z \in \mathbb{C}\},$$

given by the normalized power series

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \quad ; \quad z \in \mathbb{U} \quad ,$$

where $a_k$ is a complex number .

The class of univalent functions in $\mathcal{A}$ normalized with the conditions

$f(0) = f'(0) - 1 = 0$ was represented by $\mathcal{S}$ .

Since each function which belongs to the class has an inverse, we can easily calculate

$$(f^{-1}(w))' = \frac{1}{f'(z)} \; .$$

Therefore, we conclude the $f^{-1}$ is analytic So, $f^{-1}$ can also be displayed in the form of a power series as follows:

$$f^{-1}(w) = w - a_2\, w^2 + (2a_2^2 - a_3)w^3 - \cdots \quad ,$$

where $\quad f^{-1}\big(f(z)\big) = z \quad (z \in \mathbb{U}\,),$

$$f\left(f^{-1}(w)\right) = w \left(w \in \mathbb{U}, |w| < r_0(f); r_0(f) > \frac{1}{4}\right).$$

A function is called biunivalent in open unit disk $\mathbb{U}$, if $f$ and $f^{-1}$ are univalent in open unit disk $\mathbb{U}$.

$\Sigma$ is considered a symbol of the class of biunivalent functions in $\mathbb{U}$.(Srivastava et al., 2010)

The class of functions which are analytic in $\mathbb{U}$ and satisfying the following conditions $Re\ p(z) > 0$ and

$$p(z) = 1 + p_1(z) + p_2^2(z) + \cdots, \quad z \in \mathbb{U},$$

denoted by $\wp$.

If the function $f$ *and* $g$ are analytic in $\mathbb{U}$ ,then we say $f$ is subordinate to $g$ or $g$ is said to be superordinate to $f$ in $\mathbb{U}$ ,written as $f \prec g$ *or* $f(z) \prec g(z)$ if there exists a Schwarz function $v(z)$ analytic in $\mathbb{U}$ , *with* $|v(z)| < 1$ , So that

$$f(z) = g\big(v(z)\big)\ and\ \ z \in \mathbb{U}.$$

In particular, If the function $g$ is univalent in $\mathbb{U}$ then the subordination $f \prec g$ is equivalent to $f(0) = g(0)$ *and* $f(\mathbb{U}) = g(\mathbb{U})$ . (Miller & Mocanu, 2000)

The Hadamard product (or convolution) of two analytic functions $f$ and $g$, where $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$; $(z \in \mathbb{U})$,

denoted by $f * g$ is defined by $(f * g)(z) = f(z) * g(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k$ $(z \in \mathbb{U})$.

And by using the Hadamard product, (Amer & Darus, 2011) have recently introduced a new generalized derivative operator**.**

**Definition 1:**

For $f \in \mathcal{A}$ the operator $I^m(\lambda_1, \lambda_2, \ell, n)$ is defined by $I^m(\lambda_1, \lambda_2, \ell, n): \mathcal{A} \longrightarrow \mathcal{A}$ where

$$I^m(\lambda_1, \lambda_2, \ell, n) f(z) = \varphi^m(\lambda_1, \lambda_2, \ell)(z) * R^n f(z) \quad (z \in \mathbb{U}), \quad \rightarrow \ (*)$$

where $m \in N_0 = \{0,1,2, \ldots\ldots\}$ *and* $\lambda_2 \geq \lambda_1 \geq 0$, $\ell \geq 0$, and $R^n f(z)$ denotes the Ruseheweyh derivative operator and given by

$$R^n f(z) = z + \sum_{k=2}^{\infty} c(n,k)\ a_k b_k z^k \ ; \ (n \in N_0, z \in \mathbb{U}),$$

where $c(n,k)=\frac{(n+1)_{k-1}}{(1)_{k-1}}$.

If $f(z)=z+\sum_{k=2}^{\infty}a_kz^k$, then we easily find from the equality $(*)$ that

$$I^m(\lambda_1,\lambda_{2,}\ell,n)f(z)=z+\sum_{k=2}^{\infty}\frac{(1+\lambda_1(k-1)+\ell)^{m-1}}{(1+\ell)^{m-1}\left(1+\lambda_2(k-1)\right)^m}c(n,k)a_kz^k,$$

where $n,m\in N_0=\{0,1,2,\dots\dots\}$ *and* $\lambda_2\geq\lambda_1\geq 0,\ \ell\geq 0.$

Special cases of this operator includes :

- The Ruscheweyh derivative operator(Ruscheweyh, 1975) in the cases :
$$I^1(\lambda_1,0,l,n)\equiv I^1(\lambda_1,0,0,n)\equiv I^1(0,0,l,n)\equiv I^0(0,\lambda_2,0,n)\equiv I^0(0,0,0,n)\equiv I^{m+1}(0,0,l,n)$$
$$\equiv I^{m+1}(0,0,0,n)\equiv\mathbb{R}^n.$$
- The Salagean derivative operator(Salagean, 2006) :
$$I^{m+1}(1,0,0,0)\equiv S^n.$$
- The generalized Ruscheweyh derivative operator (Shaqsi & Darus, 2008) :
$$I^2(\lambda_1,0,0,n)\equiv R_\lambda^n.$$
- The generalized Salagean derivative operator introduced by Al-Oboudi (Al-Oboudi, 2004)
$$I^{m+1}(\lambda_1,0,0,0)\equiv S_\beta^n.$$
- The generalized Al-Shaqsi and Darus derivative operator (Shaqsi & Darus, 2008)
$$I^{m+1}(\lambda_1,0,0,n)\equiv D_{\lambda,\beta}^n\ .$$
- The Al-Abbadi and Darus generalized derivative operator (Al-Abbadi & Darus, 2009)
$$I^m(\lambda_1,\lambda_2,0,n)\equiv\mu_{\lambda_1,\lambda_2}^{n,m}\ .$$
- And finally the catas derivative operator (Catas & Borsa, 2009)
$$I^m(\lambda_1,0,l,n)\equiv I^m(\lambda_1,\beta,l).$$

Using simple computation one obtains the next result

$$(\ell+1)I^{m+1}(\lambda_1,\lambda_{2,}\ell,n)f(z)=\quad(1+\ell-\lambda_1)[I^m(\lambda_1,\lambda_{2,}\ell,n)*\varphi^1(\lambda_1,\lambda_2,\ell)(z)]f(z)+\lambda_1z\left[I^m\left(\lambda_1,\lambda_{2,}\ell,n\right)*\varphi^1(\lambda_1,\lambda_2,\ell)(z)\right]'\ ,$$

where $(z\in\mathbb{U})$ and $\varphi^1(\lambda_1,\lambda_2,\ell)(z)$ analytic function .

**Definition 2**

A function $f(z) = z + \sum_{k=2}^{\infty} a_k z^k\ ;\ z \in \mathbb{U}$ is said to be in the class $\mathcal{H}_{\Sigma}^{q}(\lambda, \beta)$ if

$$f \in \Sigma\, , Re\left\{(1-\lambda)\frac{f(z)}{z} + \lambda I^m f(z)\right\} > \beta,$$

$$Re\left\{(1-\lambda)\frac{g(w)}{z} + \lambda I^m g(w)\right\} > \beta,$$

where

$w = f(z), g = f^{-1}; (q \in (0,1) 0 \leq \lambda \leq 1, 0 \leq \beta < 1), and\ z\, , w \in \mathbb{U}$ (Rahmatan et al., 2022)

**Lemma 1**

If $\in \wp$ , then $|p_k| \leq 2\ for\ each\ k$ . (Goodman, 1983)

**Definition 3** .

A function $f(z)$ of the form (1.1) is said to be in the class $\mathcal{H}_{\Sigma}^{q}(\lambda, \alpha)$ if

$$f \epsilon\, \Sigma\, , |\, arg\{\ (1-\lambda)\frac{f(z)}{z} + \lambda\, I^m\, f(z)\}\, |\ < \frac{\alpha\pi}{2}$$

$$|\, arg\{\ (1-\lambda)\frac{g(w)}{z} + \lambda\, I^m\, g(w)\}| < \frac{\alpha\pi}{2}\ ,$$

where $w = f(z), g = f^{-1}(q \in (0,1)\ 0 \leq \lambda \leq 1\, , 0 \leq \alpha \leq 1) and\ z, w \in \mathbb{U}$. (Rahmatan et al., 2022)

**Lemma 2**

*If* $p \in \wp$ *, then* $2p_2 = p_1^2 + (4 - p_1^2)x$ *for some* $x$ *with* $|x| < 1$.

In the meantime, we obtain the bounds of certain functional of coefficients for

$f \in \mathcal{H}_{\Sigma}^{q}(\lambda, \beta)$ . (Goodman, 1983)

**Literature review**

**Theorem 1.**

*If* $f \in \mathcal{H}_{\Sigma}^{q}(\lambda, \beta)$ *. and* $\in \mathbb{R}$ *, then*

$$|a_3-\gamma a_2^2|\leq\begin{cases}\dfrac{2\delta}{\eta+k}, & |h(\gamma)|<\dfrac{1}{\eta+k}\quad ,\\ 2\delta\, h(\gamma) & |h(\gamma)|>\dfrac{1}{\eta+k}\quad ,\end{cases}$$

where

$$|h(\gamma)|=\frac{(1-\gamma)}{\eta+k}\quad ;\quad \mathcal{K}=\frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^{m}}.\frac{(n+1)_2}{(1)_2}\ ,\ \ \eta=1-\lambda\ and\ \ \delta=1-\beta\ .$$

**Proof :**

from definition for the class $\mathcal{H}_{\Sigma}^{q}(\lambda,\beta)$

$$\eta\frac{f(z)}{z}+\lambda\mathrm{I}^{m}f(z)=\beta+\delta\mathrm{P(z)}\qquad ,\quad z\in\mathbb{U}\ ,\qquad\longrightarrow\ (1)$$

$$\eta\frac{g(w)}{w}+\lambda\mathrm{I}^{m}g(z)=\beta+\delta\mathrm{q(z)}\ ;\quad z\in\mathbb{U}\quad and\quad g=f^{-1}\quad\longrightarrow\ (2)$$

respectively, where

$$p(z)=1+p_1(z)+p_2z^2+\ldots\ldots,$$

$$q(w)=1+q_1(w)+q_2w^2+\cdots,\quad for\ z,w\ \in\mathbb{U}.$$

Now, equating the coefficients in (1) and (2) , we see that

$$\eta\ \mathrm{p(z)}+\lambda\left(\mathrm{z}+\sum_{\mathrm{k=2}}^{\infty}\frac{(1+\lambda_1(\mathrm{k}-1)+\ell)^{m-1}}{(1+\ell)^{m-1}\big(1+\lambda_2(\mathrm{k}+1)\big)^{m}}\ .\mathrm{c(n\,,k)}a_{\mathrm{k}}\mathrm{z}^{\mathrm{k}}\right)=\beta+\delta\mathrm{p}\ .$$

Where k=2

$$\Longrightarrow a_2\left(\eta+\frac{\lambda(1+\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+\lambda_2)^{m}}\ .\frac{(n+1)_1}{(1)_1}\right)=\delta p_1\ .\qquad\longrightarrow(3)$$

From (1) where k =3

$$\eta a_3+\lambda\left(\frac{(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1-2\lambda_2)^{m}}\ .\frac{(n+1)_2}{(1)_2}\right)a_3=\delta p_2$$

$$\Longrightarrow\quad a_3\left(\eta+\frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^{m}}\ .\frac{(n+1)_2}{(1)_2}\right)a_3=\delta p_2\ .\qquad\longrightarrow\ (4)$$

From (2) where k =2

$$-\eta a_2+a_2\lambda\left(\frac{(1+\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-}1+\lambda_2)^m}.\frac{(n+1)_1}{(1)_1}\right)=\delta q_1$$

$$\Rightarrow \qquad -a_2\left(\eta+\lambda\frac{(1+\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+\lambda_2)^m}.\frac{(n+1)_1}{(1)_1}\right)=\delta q_1 . \qquad \longrightarrow \quad (5)$$

From (2) where k =3

$$\eta(2a_2^2-a_3)+\lambda\frac{(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m}.\frac{(n+1)_2}{(1)_2}(2a_2^2-a_3)=\delta q_2$$

$$\Rightarrow (2a_2^2-a_3)\left(\eta+\frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m}.\frac{(n+1)_2}{(1)_2}\right)=\delta q_2. \longrightarrow (6)$$

From (3) and (5) ,we conclude that

$$p_1=-q_1 , \qquad \longrightarrow \quad (7)$$

from (3) $$a_2^2\left(\eta+\frac{\lambda(1+\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+\lambda_2)^m} . \frac{(n+1)_1}{(1)_1}\right)^2=\delta^2 {p_1}^2 , \qquad \longrightarrow \quad (8)$$

and by (5) $${a_2}^2\left(\eta+\lambda\frac{(1+\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+\lambda_2)^m}.\frac{(n+1)_1}{(1)_1}\right)^2=\delta^2 {q_1}^2 . \qquad \longrightarrow \quad (9)$$

Now from (8) and (9) we obtain

$$2a_2^2\left(\eta+\frac{\lambda(1+\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+\lambda_2)^m}.\frac{(n+1)_1}{(1)_1}\right)^2=\delta^2(p_1^2+q_1^2). \qquad \longrightarrow \quad (10)$$

Also , using (4) and (6), we obtain

$$2a_2^2\left(\eta+\frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m}.\frac{(n+1)_2}{(1)_2}\right)=\delta(p_2+q_2) . \qquad \longrightarrow \quad (11)$$

From (6)

$$2a_2^2\left(\eta+\frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m}.\frac{(n+1)_2}{(1)_2}\right)-a_3\left(\eta+\frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m}.\frac{(n+1)_2}{(1)_2}\right)=\delta\, q_2,$$

and from (4) we obtain that

$$2a_3\left(\eta+\frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m}.\frac{(n+1)_2}{(1)_2}\right)-2a_2^2\left(\eta+\frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m}.\frac{(n+1)_2}{(1)_2}\right)$$
$$=\delta(p_2-q_2)\ ,\qquad \longrightarrow (12)$$

by using (4) with (6)

$$2a_2^2\left(\eta+\frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m}.\frac{(n+1)_2}{(1)_2}\right)-\frac{\delta p_2}{\left(\eta+\frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m}.\frac{(n+1)_2}{(1)_2}\right)}$$
$$.\left(\eta+\frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m}.\frac{(n+1)_2}{(1)_2}\right)=\ \delta\, q_2,$$

$$\Longrightarrow\quad 2a_2^2\left(\eta+\frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m}.\frac{(n+1)_2}{(1)_2}\right)=\delta(p_2+q_2).\longrightarrow (13)$$

Let $\mathcal{K}=\frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m}.\frac{(n+1)_2}{(1)_2}$

$$2a_3(\eta+\mathcal{K})=\delta(p_2-q_2)+2a_2^2(\eta+\mathcal{K})$$

$$\Longrightarrow\qquad a_3=\frac{\delta}{2}\frac{(p_2-q_2)}{\eta+\mathcal{K}}+a_2$$

$$a_3-\gamma a_2^2=\frac{\delta}{2}\frac{(p_2-q_2)}{\eta+\mathcal{K}}+a_2-\gamma a_2^2$$

$$\Longrightarrow\qquad |a_3-\gamma a_2^2|=|a_2^2(1-\gamma)+\frac{\delta}{2}\frac{(p_2-q_2)}{\eta+\mathcal{K}}|=|\frac{\delta}{2}(\frac{(p_2-q_2)}{\eta+\mathcal{K}}+\frac{2}{\delta}a_2^2(1-\gamma)|\ ,$$

from (13) $|a_3-\gamma a_2^2|=|\frac{\delta}{2}(\frac{(p_2-q_2)}{\eta+\mathcal{K}}+\frac{2}{\delta}(\frac{\delta}{2})\left(\frac{p_2+q_2}{\eta+\mathcal{K}}\right)(1-\gamma)|$

$$|a_3-\gamma a_2^2|=|\frac{\delta}{2}(\frac{(p_2-q_2)}{\eta+\mathcal{K}}+\left(\frac{p_2+q_2}{\eta+\mathcal{K}}\right)(1-\gamma)|$$

$$|a_3-\gamma a_2^2|=|\frac{\delta}{2}(\frac{(p_2-q_2)}{\eta+\mathcal{K}}+(p_2+q_2)\,h(\gamma)|$$

$$-(\frac{\delta}{2}(\frac{(p_2-q_2)}{\eta+\mathcal{K}}+(p_2+q_2)\,h(\gamma))\le|a_3-\gamma a_2^2|\ \le(\frac{\delta}{2}(\frac{(p_2-q_2)}{\eta+\mathcal{K}}+(p_2+q_2)\,h(\gamma)).$$

**Case 1**

$$|a_3-\gamma a_2^2|\ \le\ (\frac{\delta}{2}(\frac{(p_2-q_2)}{\eta+\mathcal{K}}+(p_2+q_2)\,h(\gamma))$$

$$\Rightarrow \qquad \frac{2}{\delta}\left(\frac{|a_3-\gamma a_2^2|}{(p_2+q_2)}\right)-\frac{(p_2-q_2)}{\eta+\mathcal{K}}\cdot\frac{1}{(p_2+q_2)} \leq h(\gamma),$$

where $|h(\gamma)| < \frac{1}{\eta+\mathcal{K}}$

$$\Rightarrow \qquad \frac{2}{\delta}\left(\frac{|a_3-\gamma a_2^2|}{(p_2+q_2)}\right)-\frac{(p_2-q_2)}{\eta+\mathcal{K}}\cdot\frac{1}{(p_2+q_2)} \leq \frac{1}{\eta+\mathcal{K}}$$

$$\Rightarrow \qquad \frac{2}{\delta}(\eta+\mathcal{K})|a_3-\gamma a_2^2|-(p_2-q_2) \leq (p_2+q_2),$$

by using ( Lemma 1)

$$\Rightarrow \qquad \frac{2}{\delta}(\eta+\mathcal{K})|a_3-\gamma a_2^2|-(p_2-q_2) \leq 4$$

$$\Rightarrow \qquad \frac{2}{\delta}(\eta+\mathcal{K})|a_3-\gamma a_2^2|-4 \leq (p_2-q_2)$$

$$\Rightarrow \qquad \frac{2}{\delta}(\eta+\mathcal{K})|a_3-\gamma a_2^2|-4 \leq 0$$

$$\Rightarrow \qquad |a_3-\gamma a_2^2| \leq \frac{4\delta}{2(\eta+\mathcal{K})} \leq \frac{2\delta}{(\eta+\mathcal{K})}$$

$$\therefore \qquad |a_3-\gamma a_2^2| \leq \frac{2\delta}{(\eta+\mathcal{K})} \quad ; \ |h(\gamma)| < \frac{1}{\eta+\mathcal{K}}\,.$$

**Case 2**

$$-\left(\frac{\delta}{2}\left(\frac{(p_2-q_2)}{\eta+\mathcal{K}}+(p_2+q_2)\,h(\gamma)\right)\right) \leq |a_3-\gamma a_2^2|$$

$$|a_3-\gamma a_2^2| \geq -\left(\frac{\delta}{2}\left(\frac{(p_2-q_2)}{\eta+\mathcal{K}}+(p_2+q_2)h(\gamma)\right)\right)$$

$$\frac{2}{\delta}|a_3-\gamma a_2^2|(\eta+\mathcal{K}) \geq -(p_2-q_2)-(p_2+q_2)\,h(\gamma)\,(\eta+\mathcal{K})$$

$$\because |h(\gamma)| = \frac{(1-\gamma)}{\eta+\mathcal{K}}$$

$$\Rightarrow \qquad \frac{2}{\delta}|a_3-\gamma a_2^2|(\eta+\mathcal{K}) \geq (q_2-p_2)-(p_2+q_2)\,\frac{(1-\gamma)}{\eta+\mathcal{K}}\,(\eta+\mathcal{K})$$

$$\Rightarrow \qquad \frac{2}{\delta}|a_3-\gamma a_2^2|(\eta+\mathcal{K}) \geq (q_2-p_2)-(p_2+q_2)\,(1-\gamma)$$

$\Rightarrow \quad \frac{2}{\delta}|a_3-\gamma a_2^2|(\eta+\mathcal{K}) \geq q_2-p_2-p_2+p_2\gamma-q_2+q_2\gamma$

$\Rightarrow \quad \frac{2}{\delta}|a_3-\gamma a_2^2|(\eta+\mathcal{K}) \geq -2p_2+p_2\gamma+q_2\gamma$

$\Rightarrow \quad \frac{2}{\delta}|a_3-\gamma a_2^2|(\eta+\mathcal{K}) \geq -2p_2+\gamma(p_2+q_2)$ ,

by using( Lemma 1) we obtain

$\Rightarrow \quad \frac{2}{\delta}|a_3-\gamma a_2^2|(\eta+\mathcal{K}) \geq -4+\gamma(4)$

$\Rightarrow \quad \frac{2}{\delta}|a_3-\gamma a_2^2|(\eta+\mathcal{K}) \geq -4(1-\gamma)$

$\Rightarrow \quad |a_3-\gamma a_2^2| \geq \frac{-2\delta(1-\gamma)}{(\eta+\mathcal{K})}$

$\Rightarrow \quad |a_3-\gamma a_2^2| \geq -2\delta\, h(\gamma)$

$\Rightarrow \quad |a_3-\gamma a_2^2| \leq 2\delta\, h(\gamma) \quad ; \quad |h(\gamma)| > \frac{1}{\eta+\mathcal{K}}$

$$\therefore \quad |a_3-\gamma a_2^2| \leq \begin{cases} \frac{2\delta}{\eta+\mathcal{K}} & ; \ |h(\gamma)| < \frac{1}{\eta+\mathcal{K}} \quad , \\ 2\delta\, h(\gamma) & ; \ |h(\gamma)| > \frac{1}{\eta+\mathcal{K}} \quad . \end{cases} \qquad \blacksquare$$

**Corollary 1 .**

*If* a function $f(z)=z+\sum_{k=2}^{\infty} a_k z^k \quad ; \quad (z \in \mathbb{U})$ belongs to the class $\mathcal{H}_{\Sigma}^{q}(1,\beta)$ and $\in \mathbb{R}$ , $\lambda=1$ ,$n, \lambda_1, l=0$ , then

$$|a_3-\gamma a_2^2| \leq \begin{cases} \frac{2\delta}{\mathcal{K}} \, , & |h(\gamma)| < \frac{1}{k} \\ 2\delta\, h(\gamma) & |h(\gamma)| > \frac{1}{k} \end{cases} \quad ,$$

where $|h(\gamma)| = \frac{(1-\gamma)}{\mathcal{K}}$ ; $\mathcal{K} = \frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^{m}} \cdot \frac{(n+1)_2}{(1)_2}, \delta = 1-\beta$.(Rahmatan et al., 2022**)**

**Theorem 2:**

If $f \in \mathcal{H}_{\Sigma}^{q}(\lambda,\beta)$ and $\mu \epsilon \mathbb{C}$ , then

$$|a_3-\mu a_2^2| \leq \begin{cases} \dfrac{4\delta}{\eta+\mathcal{K}} & ,|1-\mu|\epsilon[0,\xi) \quad , \\ \\ \dfrac{4\delta|1-\mu|}{(\eta+R)^2} & ,|1-\mu|\epsilon[\xi,\infty), \end{cases}$$

where : $R = \frac{\lambda(1+\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+\lambda_2)^m} \cdot \frac{(n+1)_1}{(1)_1}$ , $\mathcal{K} = \frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m} \cdot \frac{(n+1)_2}{(1)_2}$ ,

$\xi = \frac{(\eta+R)^2}{\delta(\eta+\mathcal{K})}$ , $\eta = 1-\lambda$ and $\delta = 1-\beta$.

**Proof :**

From (12)

$$2a_3 \left(\eta + \frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m} \cdot \frac{(n+1)_2}{(1)_2}\right) - 2a_2^2\left(\eta + \frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m} \cdot \frac{(n+1)_2}{(1)_2}\right) = \delta(p_2 - q_2)$$

$$2a_3 \ (\eta+\mathcal{K}) - 2a_2^2(\eta+\mathcal{K}) = \delta(p_2 - q_2)$$

$$\Rightarrow \qquad a_3 = \frac{\delta(p_2-q_2)}{2(\eta+\mathcal{K})} + a_2^2 \qquad \longrightarrow (14)$$

$$\Rightarrow (a_3-\mu a_2^2) = a_2^2 + \frac{\delta(p_2-q_2)}{2(\eta+\mathcal{K})} - \mu a_2^2 = \frac{\delta(p_2-q_2)}{(\eta+\mathcal{K})} + (1-\mu)a_2^2,$$

$\because \quad 2a_2^2 = \frac{\delta^2(p_1^2+q_1^2)}{2(\eta+R)^2}$, from (10) *and* $p_1 = -q_1$ ,

$$\Rightarrow \qquad (a_3-\mu a_2^2) = (1-\mu)\frac{\delta^2 p_1^2}{(\eta+R)^2} + \frac{\delta(p_2-q_2)}{2(\eta+\mathcal{K})},$$

by using( lemma 2), we obtain

$$(a_3-\mu a_2^2) = (1-\mu)\frac{\delta^2 p_1^2}{(\eta+R)^2} + \frac{\delta(4-p_1^2)}{2(\eta+\mathcal{K})}\ (x-y).$$

Let $p_1 = p$ ; $p\epsilon[0,2]$.

By triangle inequality and letting $X = |x| \leq 1,\ Y = |y| < 1$, we obtain

$$(a_3 - \mu a_2^2) = (1-\mu)\frac{\delta^2 P^2}{(\eta+R)^2} + \frac{\delta(4-p^2)}{2(\eta+\mathcal{K})}\ (x-y)$$

$$|a_3 - \mu a_2^2| \le (1-\mu)\frac{\delta^2 P^2}{(\eta+R)^2} + \frac{\delta(4-p^2)}{2(\eta+\mathcal{K})}\ (X+Y) = F(X,Y),$$

for X,Y ∈ [0,1]:

$$\max\{F(X,Y) = F(1,1) = \frac{|1-\mu|\,\delta^2 P^2}{(\eta+R)^2} + \frac{\delta(4-p^2)2}{2(\eta+\mathcal{K})}$$

$$= \frac{\delta^2}{(\eta+R)^2}\{|1-\mu| + \frac{(4-p^2)(\eta+R)^2}{P\delta^2(\eta+\mathcal{K})}\}\ p^2$$

$$= \frac{\delta^2}{(\eta+R)^2}\left\{|1-\mu| - \frac{p^2(\eta+R)^2}{\delta p^2(\eta+\mathcal{K})} + \frac{(4)(\eta+R)^2}{P^2\delta(\eta+\mathcal{K})}\right\}\ p^2$$

$$= \frac{\delta^2}{(\eta+R)^2}\left\{|1-\mu| - \frac{(\eta+R)^2}{\delta(\eta+\mathcal{K})}\right\}p^2 + \frac{4\delta}{(\eta+\mathcal{K})} = H \qquad \longrightarrow (15)$$

; H is function in $\mathbb{U}$ and for $p \in [0,2]$,

$$H'(p) = \frac{2\delta^2}{(\eta+R)^2}\left\{|1-\mu| - \frac{(\eta+R)^2}{\delta(\eta+\mathcal{K})}\right\}p\ ,$$

$H'(p)$ has a critical point $H'(p) = 0$ at $p = 0$

$\Rightarrow\ H'(p) < 0$ , $|1-\mu| \in [0, \frac{-(\eta+R)^2}{\delta(\eta+\mathcal{K})}]$ ,

let $\xi = \frac{(\eta+R)^2}{\delta(\eta+\mathcal{K})}$ then $H'(p) < 0$ ; $|1-\mu| \in [0,\xi)$,

H is a strictly decreasing function of $|1-\mu|$.

Now , from (15)

$$\max\{H\colon p \in [0,2]\} = H(0) = \frac{4\delta}{\eta+\mathcal{K}} \quad ; \quad |1-\mu| \epsilon [0,\xi) \quad . \quad \to \quad (16)$$

And since $H'(p) > 0$ ; $|1-\mu| \in [\xi,\infty)$

$$\max\{H : p \in [0,2]\} = H(2) = \frac{4\delta|1-\mu|}{(\eta+R)^2} \qquad \to \quad (17)$$

; H is an strictly increasing function of $|1-\mu|$

$$\therefore \quad |a_3-\mu a_2^2| \le \begin{cases} \dfrac{4\delta}{\eta+\mathcal{K}} & ,|1-\mu|\epsilon[0,\xi), \\ \\ \dfrac{4\delta|1-\mu|}{(\eta+R)^2} & ,|1-\mu|\epsilon[\xi,\infty), \end{cases} \qquad \blacksquare$$

**Corollary 2**

*If* a function $f(z)=z+\sum_{k=2}^{\infty}a_k z^k$ ; $(z\in\mathbb{U})$ belongs to the class $f\in\mathcal{H}_{\Sigma}^{q}(1,\beta)$ and $\mu\,\epsilon\,\mathbb{C}$ , $\lambda=1$ , $n,\lambda_1,l,=0$. then

$$|a_3-\mu a_2^2| \le \begin{cases} \dfrac{4\delta}{\mathcal{K}} & ,|1-\mu|\epsilon[0,\xi) \\ \\ \dfrac{4\delta|1-\mu|}{(R)^2} & ,|1-\mu|\epsilon[\xi,\infty), \end{cases}$$

Where : $\xi=\frac{(R)^2}{\delta(k)}$ , $\delta=1-\beta$ ,$R=\frac{\lambda(1+\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+\lambda_2)^m}\cdot\frac{(n+1)_1}{(1)_1}$ *and*

$\mathcal{K}=\frac{\lambda(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m}\cdot\frac{(n+1)_2}{(1)_2}$ . (Rahmatan et al., 2022)

**Theorem 3 :**

If a function of the form $f(z)=z+\sum_{k=0}^{\infty}a_k z^k$ is in the class $\mathcal{H}_{\Sigma}^{q}(\lambda,\alpha)$ , then:

$$|a_2| \le \frac{2\alpha}{\sqrt{2(\alpha\eta+\alpha\lambda\mathcal{K}+(1-\alpha)(\eta+\lambda R)^2)}},$$

$$|a_3| \le \frac{4\alpha^2}{(\eta+\lambda R)^2}+\frac{2\alpha}{(\eta+\lambda\mathcal{K})} \quad ,$$

where $\eta=1-\lambda$, $R=\frac{(1+\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+\lambda_2)^m}\cdot\frac{(n+1)_1}{(1)_1}$ and $\mathcal{K}=\frac{(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^m}\cdot\frac{(n+1)_2}{(1)_2}$.

**Proof**

It follows from (Definition 3) that

$$\eta\frac{f(z)}{z}+\lambda I^m f(z)=[p(z)]^{\alpha} \quad , \qquad \to (20)$$

$$\eta \frac{g(w)}{z} + \lambda I^m g(w) = [q(w)]^\alpha \quad , \qquad \rightarrow (21)$$

where $p(z) = 1 + p_1(z) + p_2 z^2 + \ldots\ldots,$

$$q(w) = 1 + q_1(w) + q_2 w^2 + \cdots, \quad for \quad z, w \in \mathbb{U}, \quad \rightarrow (22)$$

are in $\wp$ .

Now , equating the coefficients in (20) and (21), we obtain where $k = 2$ in (20)

$$a_2\left(\eta + \frac{\lambda(1+\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+\lambda_2)^m} \cdot \frac{(n+1)_1}{(1)_1}\right) = \alpha p_1$$

$$\Rightarrow \qquad a_2(\eta + \lambda\, R) = \alpha p_1 \ , \qquad \rightarrow (23)$$

from (21) where $k = 2$

$$\Rightarrow \qquad -a_2(\eta + \lambda R) = \alpha q_1$$

$\because a_2 = p_1(z) \quad \Rightarrow \quad -a_2(\eta + \lambda R) = \alpha q_1 \quad , \qquad \rightarrow (24)$

from(21) where k = 3

$$a_3(\eta + \lambda\mathcal{K}) = \alpha p_2 + \frac{\alpha(\alpha-1)}{2} p_1^2 \ , \qquad \rightarrow (25)$$

and from (21)we obtain

$$2(a_2^2 - a_3)(\eta + \lambda\mathcal{K}) = \alpha q_2 + \frac{\alpha(\alpha-1)}{2} q_1^2 \ , \quad \rightarrow (26)$$

now ,from (23) and (24),we deduce that

$$\alpha p_1 = -\alpha q_1 \ \Rightarrow p_1 = -q_1 \quad , \qquad \rightarrow (27)$$

from (23) $\quad a_2^2(\eta + \lambda\, R)^2 = \alpha^2 {p_1}^2 \qquad \rightarrow (28)$

and (24) $\quad a_2^2(\eta + \lambda\, R)^2 = \alpha^2 {q_1}^2 \qquad \rightarrow (29)$

from (28) and (29)

$$2a_2^2(\eta + \lambda\, R)^2 = \alpha^2(p_1^2 + {q_1}^2) \qquad \rightarrow (30)$$

and from (26)

$$2\,a_2^2(\eta+\lambda\mathcal{K})-a_3(\eta+\lambda\mathcal{K})=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1$$

$$\Rightarrow\quad 2\,a_2^2(\eta+\lambda\mathcal{K})-\alpha p_2-\frac{\alpha(\alpha-1)}{2}p_1^2=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1$$

$$\Rightarrow\quad 2\,a_2^2(\eta+\lambda\mathcal{K})=\alpha(p_2+q_2)-\frac{\alpha(\alpha-1)}{2}(p_1^2+q_1^2),$$

from (30),we obtain

$$\Rightarrow\quad 2\,a_2^2(\eta+\lambda\mathcal{K})=\alpha(p_2+q_2)-\frac{\alpha(\alpha-1)}{2}\left(\frac{2a_2^2(\eta+\lambda\,R)^2}{\alpha^2}\right)$$

$$\Rightarrow\quad 2\,a_2^2(\eta+\lambda\mathcal{K})=\alpha(p_2+q_2)+\frac{(\alpha-1)}{\alpha}\,(\eta+\lambda\,R)^2a_2^2\quad\rightarrow(31)$$

$$\Rightarrow\qquad a_2^2\left((\eta+\lambda\mathcal{K})-\frac{(\alpha-1)}{\alpha}\,(\eta+\lambda\,R)^2\right)=\alpha(p_2+q_2)$$

$$\Rightarrow\qquad a_2^2(2(\alpha\eta+\lambda\alpha\mathcal{K})+(1-\alpha)\,(\eta+\lambda\,R)^2)=\alpha^2(p_2+q_2)$$

$$\Rightarrow\qquad a_2^2=\frac{\alpha^2(p_2+q_2)}{2(\alpha\eta+\lambda\alpha\mathcal{K})+(1-\alpha)(\eta+\lambda\,R)^2}\ ,$$

by using (lemma 3) ,we obtain

$$\Rightarrow\qquad a_2^2\le\frac{4\alpha^2}{2(\alpha\eta+\lambda\alpha\mathcal{K})+(1-\alpha)(\eta+\lambda\,R)^2}$$

$$\Rightarrow\qquad |\,a_2\,|\le\frac{2\alpha}{\sqrt{2(\alpha\eta+\lambda\alpha\mathcal{K})+(1-\alpha)(\eta+\lambda\,R)^2}}\quad,\qquad\rightarrow(32)$$

now, from (26) $\quad 2a_2^2(\eta+\lambda\mathcal{K})-a_3(\eta+\lambda\mathcal{K})=\alpha q_2+\frac{\alpha(\alpha-1)}{2}q_1^2\ ,$

and $\quad a_3(\eta+\lambda\mathcal{K})=\alpha p_2+\frac{\alpha(\alpha-1)}{2}p_1^2\ ,$

we get the equation

$$2a_2^2(\eta+\lambda\mathcal{K})-2a_3(\eta+\lambda\mathcal{K})=\alpha q_2-\alpha p_2+\frac{\alpha(\alpha-1)}{2}q_1^2-\frac{\alpha(\alpha-1)}{2}p_1^2$$

$$2a_3(\eta+\lambda\mathcal{K})-2a_2^2(\eta+\lambda\mathcal{K})=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2\ )\,,$$

from (30) we obtain that

$$2a_3(\eta+\lambda\mathcal{K})-\frac{\alpha^2(p_1^2+{q_1}^2)}{(\eta+\lambda\, R)^2}(\eta+\lambda\mathcal{K})=\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2\ )$$

$$2a_3(\eta+\lambda\mathcal{K})=\frac{\alpha^2(p_1^2+{q_1}^2)\ (\eta+\lambda\mathcal{K})}{(\eta+\lambda\, R)^2}+\alpha(p_2-q_2)+\frac{\alpha(\alpha-1)}{2}(p_1^2-q_1^2\ )$$

$$\Rightarrow\qquad a_3=\frac{\alpha(\alpha-1)(p_1^2-q_1^2\ )}{2.2(\eta+\lambda\mathcal{K})}+\frac{\alpha(p_2-q_2)}{2\ (\eta+\lambda\mathcal{K})}+\frac{\alpha^2(p_1^2+{q_1}^2)}{(\eta+\lambda\, R)^2},$$

$\because\quad p_1=-q_1$ from (27)

$$\Rightarrow\qquad a_3=\frac{\alpha(p_2-q_2)}{2\ (\eta+\lambda\mathcal{K})}+\frac{\alpha^2(p_1^2+{q_1}^2)}{(\eta+\lambda\, R)^2}$$

$$\Rightarrow\qquad |\,a_3|=\left|\frac{\alpha(p_2-q_2)}{2\ (\eta+\lambda\mathcal{K})}+\frac{\alpha^2(p_1^2+{q_1}^2)}{(\eta+\lambda\, R)^2}\right|$$

$$\Rightarrow\qquad |\,a_3|\le\left|\frac{\alpha(p_2-q_2)}{2\ (\eta+\lambda\mathcal{K})}+\frac{\alpha^2(p_1^2+{q_1}^2)}{(\eta+\lambda\, R)^2}\right|\quad,$$

now ,by using (Lemma 1)

$$|\,a_3|\le\frac{4\alpha}{2\ (\eta+\lambda\mathcal{K})}+\frac{4\alpha^2}{(\eta+\lambda\, R)^2}$$

$$\Rightarrow\qquad |\,a_3|\le\frac{2\alpha}{(\eta+\lambda\mathcal{K})}+\frac{4\alpha^2}{(\eta+\lambda\, R)^2}\qquad\rightarrow(33)\quad\blacksquare$$

**Corollary 3**

*If* a function $f(z)=z+\sum_{k=0}^{\infty}a_kz^k$ belongs to the class $f\ \in\mathcal{H}_{\Sigma}^{q}(1,\alpha)$ and, $\lambda=1.\ n,\lambda_1,l,=0,$

then $|a_2|\le\frac{2\alpha}{\sqrt{2(\alpha k+(1-\alpha)(R)^2)}}$,

$$|a_3|\le\frac{4\alpha^2}{(R)^2}+\frac{2\alpha}{(k\ )}\ ,$$

where $\eta=1-\lambda$ , $R=\frac{(1+\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+\lambda_2)^{m}}.\frac{(n+1)_1}{(1)_1}$ and $k=\frac{(1+2\lambda_1+\ell)^{m-1}}{(1+\ell)^{m-1}(1+2\lambda_2)^{m}}.\frac{(n+1)_2}{(1)_2}$ . (Rahmatan et al., 2022)

These references (Alabbar et al., 2023) (Amer, 2016) (Agrawal & Sahoo, 2017) contain additional research and studies on analytic functions connected to the derivative operator and integral operator. It is advisable to consult these sources for further information and insights.

**Conclusion.**

In this paper ,we used new results are related to the class $\mathcal{H}_{\Sigma}^{q}(\lambda,\alpha)$ of analytic normalized function in $\mathbb{U}$ and obtained the bound s of the functional of coefficients and initial coefficient estimates of the initial Taylor and Maclaurin coefficients by using a generalized derivative operator . In a future paper , we may obtain the bounds determinants for another class of function .

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