On β- Continuity in Neutrosophic Topological Spaces

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ABSTRACT: Continuity is one of most important concepts in many mathematical disciplines. In some situations, general notion of continuity is replaced by sequential continuity.

In this paper we introduce and study the concepts of β-open set, β-continuous functions, then we also study the concepts of β-compact subsets and study some new notion of the neutrosophic β-compact space, neutrosophic simply $\boldsymbol{\beta}$-open set, neutrosophic simply $\boldsymbol{\beta}$-open cover, and neutrosophic simply $\boldsymbol{\beta}$-compact, neutrosophic $\boldsymbol{\beta}$- continuous function, neutrosophic $\boldsymbol{\beta g}$- continuous in neutrosophic topological spaces, and we present some definitions, properties, examples that illustrate its properties.

Keywords: Neutrosophic $\boldsymbol{\beta}$-compact, Neutrosophic simply $\boldsymbol{\beta}$-open set, Neutrosophic $\boldsymbol{\beta g}$-continuous, Neutrosophic $\boldsymbol{\beta}$-continuous function.

INTRODUCTION:

The importance of sequential continuity in mathematics and its applications in other sciences (such as computer science, information theory, biological science, dynamical systems and so on) is well known.
Abd El-Monsef et al. [2] introduced the notion of β-open sets and β-continuity in topologicalspaces. Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Levine [10] introduced the notion of semi-open sets and semi-continuity in topological spaces. Andrijevic [3] introduced a class of generalized open sets in topological spaces. Mashhour [12] introduced preopen sets in topological spaces.
Our goal in this paper is to extend these ideas to neutrosophic topological spaces. We introduce the neutrosophic β-compact, neutrosophic simply β-open set, neutrosophic βg-continuous, neutrosophic β-continuous function, and investigate some their properties.
We use standard terminology and notations for neutrosophic set theory and the theory of neutrosophic topological spaces following mainly [18, 21].

1- PRELIMINARIES

Definition 1.1

A subset A of a topological space (X, τ) is called:
1- Preopen [12] if $A \subseteq int(cl(A))$.
2- Semi open [10] if $A \subseteq cl(int(A))$.
3- Regular open [22] if $A = int(cl(A))$.
4- β-open [1] if $A \subset cl\left(int(cl(A))\right)$.

Definition 1.2 [1, 6]

A function $f: X \longrightarrow Y$ is called:
1. semi continuous if $f^{-1}(V)$ is semi open in X for each open set V of Y.
2. pre continuous if $f^{-1}(V)$ is preopen in X for each open set V of Y.

3. α -continuous if $f^{-1}(V)$ is α - open in X for each open set V of Y.
4. b-continuous if $f^{-1}(V)$ is b-open in X for each open set V of Y.
5. β-continuous if $f^{-1}(V)$ is β -open in X for each open set V of Y.

Definition 1.3 [11]

A topological space (X,τ) is called compact space if every open cover of X has a finite sub cover.

Theorem 1.1 [11]

A closed sub set of a compact space is compact.

Definition 1.4 [6]

A space X is called a β- compact space if each β- open cover of X has a finite sub cover for X.

Definition 1.5 [1]

A function $f:(X,\tau) \longrightarrow (Y,\sigma)$ is called β- open if $f(G) \in \beta O(Y)$ for every $G \in \tau$.

Definition 1.6 [11]

A topological space (X,τ) is called:
Regular if for every $A \in \tau^C, x \notin A$ then there exist $U,V \in \tau, U \cap V = \emptyset$ such that $x \in U$ & $A \subset V$.

Theorem 1.2 [14]

For a topological space (X,τ) then the following statement are equivalent:
1- X is regular (resp. almost regular).
2- For any closed set (resp. regular closed) set F and each $x \in X - F$, there exist two disjoint open U & V such that $x \in U$ & $F \subset V$.
3- For any open (resp. regular open) set V containing x in X there exist an $U \in \tau^\alpha$ such that $x \in U \subset \alpha cl(A) \subset V$.

Definition 1.7 [1]

Let f be a function from a neutrosophic topological spaces (X,τ) and (Y,S) then f is called:
i- a neutrosophic open function if $f(A)$ is a neutrosophic open set in Y for every aneutrosophic open set A in X.
ii- a neutrosophic α- open function if $f(A)$ is a neutrosophic α- open set in Y for every aneutrosophic open set A in X.
iii- a neutrosophic preopen function if $f(A)$ is a neutrosophic preopen set in Y for every aneutrosophic open set A in X.
iv- a neutrosophic semiopen function if $f(A)$ is a neutrosophic semiopen set in Y for every aneutrosophic open set A in X.

Definition 1.8 [16]

A NS C in NTS $\mathcal{U}$ is so called a neutrosophic generalized closed set and denoted by N_gCS if for any N_gOS M in $\mathcal{U}$ such that $C \subseteq M$, then $Ncl(C) \subseteq M$. Moreover, its complement is named a neutrosophic generalized open set referred to N_gOS.

Definition 1.9 [7]

Let (X,f) be NTS and B be a NS in X. Then neutrosophic generalized closure is defined as, $G\mathrm{Ncl}(B) = \bigcap\{G: G \text{ is a } GNCS \text{ in } X \text{ and } B \subseteq G\}$.

Definition 1.10 [7, 19]

A map f: X ⟶ Y is said to be:
i- neutrosophic closed (in short, NC-map) if the image of every NCS in X is a NCS in Y.
ii- neutrosohic continuous (in short, N-continuous) if inverse image of every NCS in Y is a NCS in X.
iii- generalized neutrosohic continuous (in short, GN-continuous) if inverse image of every NCS in Y is a GNCS in X.
iv- generalized neutrosohic irresolute (in short, GN-irresolute) if inverse image of every GNCS in Y is a GNCS in X.

Definition 1.11 [15]

A bijective $f: X \longrightarrow Y$ is called a neutrosophic homeomorphism if f & f^{-1} are neutrosophice continuous.

Definition 1.12 [9]

A bijective $f: X \longrightarrow Y$ is named as neutrosophic generalized homeomorphism (in short neutrosophic f-homeomorphism) if f & f^{-1} are GN-continuous.

Definition 1.13 [9]

A mapping $\eta: X \longrightarrow Y$ is generalized neutrosophic open (in short, GNO-map) if The image $\eta(R)$ is $GNOS$ in Y for every $NOSR$ in X.

Definition 1.14 [9]

A mapping $\eta: X \longrightarrow Y$ is generalized neutrosophic closed (in short, GNC-map) if the image $\eta(Q)$ is $GNCS$ in Y for every $NCSQ$ in X.

Proposition 1.1 [9]

Every NC-mapping is a GNC-mapping.

Proposition 1.2 [9]

A map $\eta: X \longrightarrow Y$ is a GNC-mapping if the image of each NOS in X is $GNOS$ in Y.

Proposition 1.3 [9]

Let $\mu: X \longrightarrow Y$ and $\lambda: Y \longrightarrow Z$ be $NTSs$, then the following hold.
i- If $(\lambda \circ \mu)$ is GNO-map and μ is N-continuous, then λ is GNO-map.
ii- If $(\lambda \circ \mu)$ is GNO-map and μ is GN-continuous, then λ is GNO-map.

Definition 1.15 [17]

Let $(\mathcal{U}, \zeta)$ and $(\mathcal{V}, \sigma)$ be $NTSs$ and $\eta: (\mathcal{U}, \zeta) \longrightarrow (\mathcal{V}, \sigma)$ be a mapping we have:
i- if for each NOS (correspondingly, NCS) K in $\mathcal{V}$, $\eta^{-1}(K)$ is neutrosophic continuous and denoted by N-continuous [20].
ii- if for each NOS(correspondingly, NCS) K in $\mathcal{V}$, $\eta^{-1}(K)$ is a $N-\alpha OS$ (correspondingly, $N-\alpha CS$) in $\mathcal{U}$, then η is known as neutrosophic α-continuous and referred to $N-\alpha$-continuous [4] .
iii- if for each NOS(correspondingly, NCS) K in $\mathcal{V}$, $\eta^{-1}(K)$ is a $N-gOS$ (correspondingly, $N-gCS$) in $\mathcal{U}$, then η is known as neutrosophic g-continuous and signified by $N-g$-continuous [8] .

Definition 1.16 [17]

Let η be a function on $NTS\ \mathcal{U}$ and valued in $TS\ \mathcal{V}$, then we named η as aneutrosophic generalized αg-continuous and shortly wrote it as $Ng\alpha g$- continuous if for each $NCS\ K$ in $\mathcal{V},,\eta^{-1}(K)$ is a $Ng\alpha gCS$ in $\mathcal{U}$.

Definition 1.17 [5]

Let f be a function from a neutrosophic topological space (X,τ_1) to neutrosophic topological space (Y,τ_2). Then f is called a neutrosophic pre continuous function if $f^{-1}(B)$ is a neutrosophic preopen in X for every neutrosophic open set B in Y.

2- NEAR COMPACTNESS and CONTINUITY

In this section we shall state some types of compact spaces and their properties.

Definition 2.1

1- A neutrosophic topological space (X,τ) is called a neutrosophic β-compact if every neutrosophic β-open cover of X has a finite subcover.

2- A neutrosophic topological space (X,τ) is called a neutrosophic simply β-

compact space if every neutrosophic simply β-open cover of X has a finite sub-cover.

Definition 2.2

A neutrosophic subset A of (X,τ) is said to be a neutrosophic simply β- compact set relative to X if every neutrosophic simply β-open cover of A has a finite sub cover.

Definition 2.3

Let f be a function from a neutrosophic topological space (X,τ_1) to a neutrosophic topological space (Y,τ_2) then f is called:

i- a neutrosophic open function if $f(A)$ is a neutrosophic open set in Y for every aneutrosophic open set A in X. [5]

ii- a neutrosophic β-open function if $f(A)$ is a neutrosophic β-open set in Y for every aneutrosophic open set A in X.

Theorem 2.1

i- Every neutrosophic β-compact space is a neutrosophic simply β-compact space.

ii- Every neutrosophic simply β-compact space is a neutrosophic compact space.

Proof:

i- Suppose that (X,τ) is a neutrosophic β-compact space. Let $B=\{U_\gamma\}$ be a neutrosophic β-open cover of X. We need to find a countable subcover of B that covers X. Since (X,τ) is a neutrosophic β-compact space, there exist a finite subcover $B^{\sim}=\{U_{\gamma_1},U_{\gamma_2},U_{\gamma_3},\dots,U_{\gamma_n}\}$ of B that covers X. We can construct a countable of B by adding each U_{γ_i} for $i=1,2,3,\dots n$, to countable set S.

Therefore, $S=\{U_{\gamma_1},U_{\gamma_2},U_{\gamma_3},\dots,U_{\gamma_n}\}$ is a countable subcover of B that covers X. Hence, every neutrosophic β-compact space is a neutrosophic simply β-compact space.

ii- Let X be a simply β-compact space, and suppose that $\{U_\gamma\}$ is an open cover of X. By the definition of neutrosophic simply β-compact for every open set U and closed set V such that $U \cap V = \emptyset$, there exists an open set B such that $U \subseteq B$ &
$B \cap V = \emptyset$. Now, let define a finite subset $\{U_{\gamma_1}, U_{\gamma_2}, U_{\gamma_3}, \ldots, U_{\gamma_n}\}$ such that it covers X. By the definitions of compactness and neutrosophic simply , , there exist a finite subcover for every open cover. This, X is compact, X satisfies the neutrosophic simply β propers.

Finally, by satisfying both neutrosophic simply β and compactness properties, it can be concluded that every neutrosophic simply β-compact space is a neutrosop- hic compact space.

Theorem 2.2

i- If $f: (X, \tau_1) \longrightarrow (Y, \tau_2)$ is a neutrosophic β-open function and (Y, τ_2) is a neutrosophic β-compact space, then (X, τ_1) is also a neutrosophic compact space.

ii- If $f: (X, \tau_1) \longrightarrow (Y, \tau_2)$ is a neutrosophic simply β-open function and (Y, τ_2) is a neutrosophic simply β-compact space, then (X, τ_1) is also a neutrosophic β-compact space.

Proof

i- Suppose that $f: (X, \tau_1) \longrightarrow (Y, \tau_2)$ is a neutrosophic β-open function, and (Y, τ_2) is a neutrosophic β-compact space, we will use the concept of a neutrosophic open cover. Let $B = \{U_i\}$ be a neutrosophic open cover of (X, τ_1). Since f is a neutrosophic β-open function, for every U_i in B, the $f^{-1}(\,U_i)$ is β- open in (X, τ_1). Now, suppose the collection $U = \{\,f^{-1}(\,U_i): U_i \in C\}$. Since f is a function from (X, τ_1) to (Y, τ_2), U is a collection of subset of X. Since B is a neutrosophic open cover of (X, τ_1), it follows that U is a collection of β-open sets in (X, τ_1). Since (Y, τ_2) is a neutrosophic β-compact space, there exists a finite subcover
$U^\sim = \{\,f^{-1}(\,U_1),\ f^{-1}(\,U_2),\ f^{-1}(\,U_3), \ldots,\ f^{-1}(\,U_n)\,\}$ of U that covers X.
Now, let $B^\sim = \{\,U_1, U_2, U_3, \ldots, U_n\}$ be the corresponding finite subcover of B. Since $U^\sim$ is a finite subcover of U, it follows that $B^\sim$is a finite subcover of B. Hence (X, τ_1) is a neutrosophic compact space.
ii- The proof of this part is similar to the above part (ii).

Theorem 2.3

i- If $f: (X, \tau_1) \longrightarrow (Y, \tau_2)$ is a neutrosophic β-continuous function, then for each neutrosophic β-compact set U in $X, f(U)$ is a neutrosophic simply β-compact set in Y.
ii- If $f: (X, \tau_1) \longrightarrow (Y, \tau_2)$ is a neutrosophic β-continuous function, then for each neutrosophic β-compact set K in $X, f(K)$ is a neutrosophic compact set in Y.

Proof

i- Suppose that $f: (X, \tau_1) \longrightarrow (Y, \tau_2)$ is a neutrosophic β-continuous function, and U is a neutrosophic β-compact set in X, since f is a neutrosophic β-continuous function for every β-open set V in Y, the $f^{-1}(V)$ is β-open in X. Now, suppos the collection $D = \{f^{-1}(V): V\ is\ a\ \beta - \text{open set in } Y\}$. This collection D consists of β-open sets in X. Since U is a neutrosophic β-compact set relative to X, there exists a finite subcover,
$D^\sim = \{\,f^{-1}(\,V_1), f(\,V_2),\ f^{-1}(\,V_3), \ldots,\ f^{-1}(\,V_n)\,\}$ of D that covers U. Now, let $C^\sim = \{V_1, V_2, V_3, \ldots, V_n\,\}$ be the corresponding finite subcovew of C. It follows that $f(U)$ is covered by the finite subcover $C^\sim$, which mesns $f(U)$ is a neutrosophic simply β- compact set in X.
ii- Proof this a part similar to i.

Proposition 2.1

Let (X, τ_1), (Y, τ_2) & (Z, τ_3) be three neutrosophic topological spaces, let $f: (X, \tau_1) \longrightarrow (Y, \tau_2)$ and $g: (Y, \tau_2) \longrightarrow (Z, \tau_3)$ be functions. If f is neutrosophic open and g is neutrosophic β-open then $g \circ f$ is neutrosophic β-open.

Proof

Suppose that, f is a neutrosophic open, g is neutrosophic β- open and prove a composition function $g \circ f$ is neutrosophic β-open. Let A be a neutrosophic β-open set in (Z, τ_3). By the definition of neutrosophic β-open set, we can represent A as $A = N_\beta(A^\sim)$, where $A^\sim$ is a regular open set in (Z, τ_3). Since g is neutrosophic β- open, we have $g^{-1}(A^\sim)$ is a neutrosophic β-open set in (Y, τ_2).
Suppose that $(g \circ f)^{-1}(A) = f^{-1}\big(g^{-1}(A)\big)$. Since f is a neutrosophic open function, then $f^{-1}\big(g^{-1}(A)\big)$ is a neutrosophic open set in (X, τ_1). Therefore, $g \circ f$ is neutrosophic β-open.

Proposition 2.2

Let (X, τ_1) and (Y, τ_2) are neutrosophic topological spaces. If $f: (X, \tau_1) \longrightarrow (Y, \tau_2)$ is a neutrosophic β-open then it is neutrosophic semi open.

Proof

Suppose that f is a neutrosophic β-open function, and let U be a neutrosophic open set in (Y, τ_2). By definition of a neutrosophic open set U can be represented as $U = N_\beta(B)$, where B is a regular open set in (Y, τ_2).
Since f is a neutrosophic β- open function, we know that $f^{-1}(B)$ is a neutrosophic β-open set in (X, τ_1) for any regular open set B in (Y, τ_2).
In particular, since B is a regular open set, we know that $f^{-1}(B)$ is a neutrosophic β- open set in (X, τ_1).
It is clear that, $f^{-1}(B)$ is a subset of $f^{-1}(U)$. Since neutrosophic β-open sets are closed under subsets, we can conclude that $f^{-1}(B)$ is a neutrosophic semi- open set in (X, τ_1). So, we have shown that for every neutrosophic open set U in (Y, τ_2), $f^{-1}(U)$, is a neutrosophic semi- open set in (X, τ_1).
Therefore, f^{-1} is a neutrosophic semi-open function.

Proposition 2.3

Let (X, τ_1) and (Y, τ_2) are neutrosophic topological spaces. If $f: (X, \tau_1) \longrightarrow (Y, \tau_2)$ is a neutrosophic β-open then it is neutrosophic preopen.

Proof

The proof is similar to the previous theorem above.

Definition 2.4

Let f be a function from a neutrosophic topological space (X, τ_1) to a neutros-ophic topological space (Y, τ_2), then f is called a neutrosophic β-continuous funct- ion if $f^{-1}(U)$ is a neutrosophic β-open set in X for every neutrosophic open set U in Y.

Proposition 2.4

Let f a function from a neutrosophic topological space (X, τ_1) to a neutrosophic topological space (Y, τ_2), satisfies the condition:

$Ncl\left(Nint\left(Ncl(f^{-1}(U))\right)\right) \subseteq f^{-1}(Ncl(U))$ for every neutrosophic set U in Y. Then f is a neutrosophic β-continuous function.

Proof

Suppose that f satisfies the given condition, and let U be a neutrosophic set in Y. By the definition of a neutrosophic set, U can be represented as
$U = Nint(Nint(U))$. Now, using the given condition, we have
$Ncl\left(Nint(Nclf^{-1}(U))\right) \subseteq f^{-1}(Ncl(U))$. Since $U = Nint(Nint(U))$, we can substitute U in the above expression to get:

$$Ncl\left(Nint\left(Ncl\left(f^{-1}\left(Nint(Nint(U))\right)\right)\right)\right) \subseteq f^{-1}\left(Ncl\left(Nint(Nint(U))\right)\right)$$

Now, observe that $Ncl\left(Nint\left(Ncl\left(f^{-1}\left(Nint(Nint(U))\right)\right)\right)\right)$ is a closed set in X. Therefore, we have:

$$Nint\left(Nint\left(f^{-1}\left(Ncl\left(Nint(Nint(U))\right)\right)\right)\right) \subseteq f^{-1}\left(Ncl\left(Nint(Nint(U))\right)\right).$$

This implies that $f^{-1}\left(Nint(Nint(U))\right)$ is a neutrosophic β-open set in X.
Since $Nint(Nint(U)) = U$, we can conclude that $f^{-1}(U)$ is a neutrosophic β- open set in X. Therefore, f is a neutrosophic β-continuous function.

Proposition 2.5

Let f be a function from a neutrosophic topological space (X, τ_1) to neutrosophc topological space (Y, τ_2). If f is neutrosophic β-continuous, then itis neutrosophic semi-continuous.

Proof

Suppose that f is neutrosophic β-continuous, and let U be a neutrosophic open set in Y. By the definition of a neutrosophic open set, U can be represented as
$U = N_\beta(B)$, where B is a regular open set in Y.
Since f is a neutrosophic β- continuous function, then for every regular open set B in Y, if $x \in f^{-1}(B)$, then there exists a regular open set C in X such that $x \in C$ & $f(C) \subseteq B$.
Now, consider that x is any point in $f^{-1}(U)$, since $x \in f^{-1}(U)$, we know that $f(x) \in U$, such that $U = N_\beta(B)$, then there exist a regular open set C in X such that $x \in C$ and $f(C) \subseteq B$. Therefore, we can choose C as a regular open neighborhood of x contained in $f^{-1}(U)$. Thus $f^{-1}(U)$ is neutrosophic semi- open set in X. Therefore, f is a neutrosophic semi- continuous function.

Proposition 2.6

Let f be a function from a neutrosophic topological space (X, τ_1) to neutrosophic topological space (Y, τ_2). If f is neutrosophic β-continuous, then it is neutrosophic pre-continuous.

Proof

Assume that f is neutrosophic β-continuous and let U be a neutrosophic pre-open set in Y. By definition of a neutrosophic preopen set, U can be represented as $U = N_\beta(B)$, where B is a regular closed set in Y.

Since f is a neutrosophic β-continuous function, then we have

$Ncl\left(Nint\left(Ncl(f^{-1}(B))\right)\right) \subseteq f^{-1}(Ncl(B))$, for every regular closed set B in Y, since $U = N_\beta(B)$ so that

$Ncl\left(Nint\left(Ncl(f^{-1}(B))\right)\right) \subseteq f^{-1}\left(Ncl\left(N_\beta(B)\right)\right)$, i.e.

$Ncl\left(Nint\left(Ncl(f^{-1}(B))\right)\right) \subseteq f^{-1}(Ncl(B))$. Since $Ncl\left(N_\beta(B)\right) = B$, we have

$Ncl\left(Nint\left(Ncl\left(f^{-1}\left(N_\beta(B)\right)\right)\right)\right) \subseteq f^{-1}(B)$. This implies that $f^{-1}\left(N_\beta(B)\right)$ is a subset of $f^{-1}(B)$. So that $Nint(f^{-1}(B))$ is a neutrosophic open set in X. Since $f^{-1}\left(N_\beta(B)\right) \subseteq Nint(f^{-1}(B))$, then $f^{-1}\left(N_\beta(B)\right)$ is a neutrosophic semi-open set in X.

Finally, since neutrosophic semi-open sets are closed under complemente, we can conclude that $\left(f^{-1}\left(N_\beta(B)\right)\right)^C$, is a neutrosophic semi-open set in X.

Since $\left(f^{-1}\left(N_\beta(B)\right)\right)^C = f^{-1}(Ncl(B))$, then $f^{-1}(Ncl(B))$ is a neutrosophic semi open set in X. Therefore, f is a neutrosophic pre-continuous function.

Result 2.1

Every neutrosophic simply β-continuous function from a $NTS\ (X, \tau_1)$ to a $NTS\ (Y, \tau_2)$ is not a neutrosophic β-continuous function in general.

Example 2.1

Let $X = \{0,1\}$ with the discrete topology $\tau_1 = \{\emptyset, \{0\}, \{1\}, \{0,1\}\}$. Let

$Y = \{0,1\}$ with the discrete topology $\tau_2 = \{\emptyset, \{0,1\}\}$.

Now let us define a function $f: (X, \tau_1) \longrightarrow (Y, \tau_2)$ as follows $(0) = 0$, $f(1) = 1$.

This function is not a N_β-continuous function because for every N_β-open set B in (Y, τ_2), $f^{-1}(B)$ may not be a N_β-open set in (X, τ_1). Since if $B = \{0\}$,

$f^{-1}(B) = \{0\}$ which is not a N_β-open set in (X, τ_1). Therefore, f is not a N_β-continuous function.

3- Neutrosophic Generalized βg- Continuous Function

In this part of this paper, the neutrosophic generalized βg-continuous function are performed and examined their fundamental features.

Definition 3.1

Let f be a function from a neutrosophic topological space X to a topological space Y, then we called f is aneutrosophic generalized βg-continuous ($N_{g\beta g}$-continuous) if for each neutrosophic closed set B in Y, $f^{-1}(B)$ is a $N_{g\beta g}\ CS$ in X.

Theorem 3.1

If $f: (X, \tau_1) \longrightarrow (Y, \tau_2)$ is a neutrosophic simply β-continuous mapping and $g: (Y, \tau_2) \longrightarrow (Z, \tau_3)$ is a neutrosophic continuous mapping, then the composition mapping, $g \circ f: (X, \tau_1) \longrightarrow (Z, \tau_3)$ is a neutrosophic simply β-continuous mapping.

Proof

Let V be a neutrosophic β-open set in (Z, τ_3). Since g is a neutrosophic β- continuous mapping, we know that $(f)^{-1}(g)^{-1}(V)$ is a neutrosophic β-open set in (X, τ_1). Appling the property of inverse images, we have

$(g \circ f)^{-1}(V) = (f)^{-1}(g)^{-1}(V)$. Therefore, $(g \circ f)^{-1}(V)$ is a neutrosophic β- open set in (X, τ_1). This proves that the composition mapping

$g \circ f: (X, \tau_1) \longrightarrow (Z, \tau_3)$ is a neutrosophic simply β-continuous mapping.

Definition 3.2

Let $f: (X, \tau_1) \longrightarrow (Y, \tau_2)$ be a map so as X and Y are neutrosophic topological spaces, then:

i- f is called a neutrosophic βg-continuous ($N_{\beta g}$-continuous) if for every NOS

(correspondingly, NCS) K in $Y, f^{-1}(K)$ is a $N_{\beta g}OS$ (correspondingly, $N_{\beta g}CS$) in X.

ii- f is called a neutrosophic gβ-continuous ($N_{g\beta}$-continuous) if every NOS (correspondingly, NCS) K in $Y, f^{-1}(K)$ is a $N_{g\beta}OS$ (correspondingly $N_{g\beta}CS$) in X.

Theorem 3.2

Let f be a function from a NTS X to a TS Y, so we have the following:

i- All neutrosophic generalized continuous (N_g-continuous) functions are neutrosophic β-generalized continuous ($N_{\beta g}$-continuous).

ii- All neutrosophic β-continuous (N_β-continuous) functions are neutrosophic generalized β-continuous ($N_{g\beta}$-continuous).

iii- All neutrosophic generalized β-continuous ($N_{g\beta}$-continuous) functions are neutrosophic β-generalized ($N_{\beta g}$-continuous).

Proof

i- All N_g-continuous functions are $N_{\beta g}$-continues.

Suppose that $f: X \longrightarrow Y$ be an N_g-continuous function, where X is a neutrosophic topological space and Y is a topological space.

We need to show that for every $N_{\beta g}$-open set B in Y, the $f^{-1}(B)$ is an $N_{\beta g}$-open set in X.

Since B is $N_{\beta g}$-open, it can be represented as $B = N_{\beta g}(B^{\sim})$, where $B^{\sim}$ is a regular open set in Y.

Now, since f is N_g-continuous, we have $f^{-1}(B^{\sim})$ is N_g-open in X. Since N_g-open sets are $N_{\beta g}$-open, $f^{-1}(B^{\sim})$ is also $N_{\beta g}$-open in X. Since $N_{\beta g}(B^{\sim}) = B$, we can conclude that $f^{-1}(B)$ is $N_{\beta g}$-open in X. Therefor, f is $N_{\beta g}$ -continuous.

ii- All N_β-continuous functions are $N_{g\beta}$-continues.

Proof this a part similar to i.

iii- All $N_{g\beta}$-continuous functions are $N_{\beta g}$-continuous.

Proof this a part similar to i.

Theorem 3.3

Let f be a function from a neutrosophic topological space X to a topological space Y, then f is a $N_{g\beta g}$-continuous function iff for each neutrosophic open set
(NOS) V in $Y, f^{-1}(V)$ is a neutrosophic generalized β-generalized open set $(N_{g\beta g}OS)$ in X.

Proof

⟹ Suppose that f is an $N_{g\beta g}$-continuous function and let V be a NOS in Y.

We need to show that $f^{-1}(V)$ is N_g-open and $N_{\beta g}$-open in X. Since f is $N_{g\beta g}$-continuous, it is also N_g-continuous. This implies that for each N_g-open set V in Y, $f^{-1}(V)$ is a N_g-open set in X.

Now, by the definition of a NOS, V can be represented as $V = N_g\left(N_{\beta g}(V^{\sim})\right)$, where $V^{\sim}$ is a regular open set in Y. Since f is $N_{g\beta g}$- continuous, for any $N_{\beta g}$-open set B in Y, we have $f^{-1}(B)$ is a $N_{g\beta g}$-open set in X.

In particular, for $B = N_g\left(N_{\beta g}(V^{\sim})\right)$, we have $f^{-1}\left(N_g\left(N_{\beta g}(V^{\sim})\right)\right)$ is $N_{g\beta g}$-open set in X. Also, observe that $f^{-1}\left(N_g\left(N_{\beta g}(V^{\sim})\right)\right)$ is a subset of $f^{-1}(V)$. This implies that $f^{-1}(V)$ is $N_{g\beta g}$- open set in X. Therefore, we have shown that $f^{-1}(V)$ is both N_g-open and $N_{\beta g}$-open in X. So it is a $N_{g\beta g}$-open set in X.

⟸ Suppose that for each NOS V in Y, $f^{-1}(V)$ is a $N_{g\beta g}$-open set in X,

Let B be a $N_{g\beta g}$-open set in Y. By the definition of $N_{g\beta g}$-open sets, B can be represented as

$B = N_g\left(N_{\beta g}(B^{\sim})\right)$, where $B^{\sim}$ is a regular open set in Y. Since $B^{\sim}$ is regular open, it is a NOS. Hence, we have $f^{-1}(B^{\sim})$ is a $N_{g\beta g}$-open set in X.

Since $N_{g\beta g}$-OS is a subset of $N_{\beta g}$- OS and N_g- OS, we have $f^{-1}(B^{\sim})$ is also a $N_{\beta g}$-OS and N_g-OS in X.

This implies that $f^{-1}(B^{\sim})$ contains a $N_{g\beta g}$-open set, and thus it is a $N_{g\beta g}$-open set in X. Hence, we have shown that for each $N_{g\beta g}$-open set B in Y, $f^{-1}(B)$ is a $N_{g\beta g}$- open set in X. Therefore, f is a $N_{g\beta g}$-continuous function.

Proposition 3.1

For all $N_{g\beta g}$-continuous functions are $N_{g\beta}$-continuous.

Proof

Suppose that $f: (X, \tau_1) \longrightarrow (Y, \tau_2)$ is $N_{g\beta g}$-continuous, let B be a N_β-open set in Y. Since B is N_β-open, it can be represented as $B = N_\beta(B^\sim)$, where $B^\sim$ is a regular open set in X, and since f is $N_{g\beta g}$-continuous, we know that for every $N_{g\beta g}$-open set C in Y, $f^{-1}(C)$, is $N_{g\beta g}$-open in X .

In particular, for $B^\sim$, which is a regular open set and thus a $N_{g\beta g}$-open set, we have that $f^{-1}(B^\sim)$ is $N_{g\beta g}$-open in X.

Since we observe that $f^{-1}(B^\sim)$ is a subset of $f^{-1}(B)$, $N_{g\beta g}$-open sets are subsets of $N_{g\beta}$- open sets, we can conclude that $f^{-1}(B^\sim)$ is also $N_{g\beta}$- open in X, then for every N_β-open set B in Y, the $f^{-1}(B^\sim)$ is $N_{g\beta}$-open in X. Therefore for all $N_{g\beta g}$- continuous functions are $N_{g\beta}$-continuous.

Remark 3.1

The reverse of the previous proposition is not valid as shown in the next example.

Example 3.1

Let, $X = \{a, b, c\}$, & $Y = \{1,2\}$, $f: X \longrightarrow Y$ be a function defined as $f(a) = 1$, $f(b) = 1$, & $f(c) = 2$.

Now, let, $g: [0,1] \longrightarrow X$, such that $g(t) = a$, for $0 \leq t < 0.5$, and $g(t) = b$, for $0.5 \leq t < 1$.

In this case, g is a $N_{g\beta}$ on X, and it is continuous, in the sense that the inverse image of any neutrosophic crisp open set in Y under the composition $f \circ g$ is a neutrosophic crisp open set in $[0,1]$.

However, the function f is not continuous in the generalized case.

If we consider a $N_{g\beta}$ on $[0,1]$, denoted as $\delta: [0,1] \longrightarrow Y$, such that $\delta(t) = 1$, for $0 \leq t < 0,5$ and $\delta(t) = 2$, for $0.5 \leq t < 1$, then the neutrosophic $\delta \circ f$ is not a neutrosophic crisp generalized continuous function on X.

Definition 3.3

Let f be a function from a neutrosophic topological space X to a topological space Y, then f is called a neutrosophic generalized βg-irresolute and wrote it as $Ng\beta g$- irresolute if for each $Ng\beta gCS$ V in Y, $f^{-1}(V)$ is a $Ng\beta gCS$ in X.

Theorem 3.4

Let f be a function from a neutrosophic topological space X to a topological space Y, then f is a $N_{g\beta g}$-irresolute function iff for each $N_{g\beta g}OS$ U in Y, $f^{-1}(U)$ is a $N_{g\beta g}OS$ in X.

Proof

$\Longrightarrow$ Suppose that f is $N_{g\beta g}$-irresolute. Let U be a $N_{g\beta g}OS$ in Y, by the definition of

a $N_{g\beta g}OS$, U can be represented as $U = N_g\left(N_{g\beta}(U^\sim)\right)$, where $U^\sim$ is a regular open set in Y.

Now, since f is $N_{g\beta g}$-irresolute, we know that for every $N_{g\beta g}$-open set B in Y, the $f^{-1}(B)$, is $N_{g\beta g}$-open in X.

In particular, for $U^{\sim}$, which is a regular open set and thus a $N_{g\beta g}$-open set in Y, we have that $f^{-1}(U^{\sim})$ is $N_{g\beta g}$-open in X.

Now, since $f^{-1}(U^{\sim})$ is a subset of $f^{-1}(U)$, and since any $N_{g\beta g}$-open sets are subsets of $N_{g\beta g}$, we can conclude that $f^{-1}(U^{\sim})$ is also a $N_{g\beta g}OS$ in X.

Therefore, $f^{-1}(U)$, is a $N_{g\beta g}OS$ in X.

$\Leftarrow$ Suppose that for each $N_{g\beta g}OS$ U in Y, $f^{-1}(U)$ is a $N_{g\beta g}OS$ in X, and let U be a $N_{g\beta g}$-open set in Y. By the definition of a $N_{g\beta g}$-open set, U can be represented as $U = N_g\left(N_{\beta g}(U^{\sim})\right)$, where $U^{\sim}$ is a regular open set in Y. c

Now, sine $f^{-1}(U^{\sim})$ is a $N_{g\beta g}OS$ in X, and any $N_{g\beta g}OS$ is a subset of $N_{g\beta g}$-open sets, we have that $f^{-1}(U^{\sim})$ is also $N_{g\beta g}$-open in X, it is observe that $f^{-1}(U^{\sim})$ is a subset of $f^{-1}(U)$. Hence, $f^{-1}(U)$ is a $N_{g\beta g}$-open in X. Therefore, f is a $N_{g\beta g}$- irresolute function.

Proposition 3.2

For all $N_{g\beta g}$-irresolute function is $N_{g\beta}$-continuous.

Proof

Suppose that f is $N_{g\beta g}$-irresolute from the space X into a space Y, and let V be a N_{β}-open set in Y, We need to show that f is a $N_{g\beta}$-continuous mapping that if a function f is a N_{β} -open set V in Y, the $f^{-1}(V)$ is a $N_{g\beta}$ in X.

Now since V is N_{β}-open, so we can represent it as $V = N_{\beta}(V^{\sim})$, where $V^{\sim}$ is a regular open set in Y, and since f is $N_{g\beta g}$-irresolute, so for every $N_{g\beta g}$ OS B in Y, the $f^{-1}(B)$, is a $N_{g\beta g}$ OS in X.

In particular, for $V^{\sim}$, which is a regular open set and thus a $N_{g\beta g}$ OS, we have that $f^{-1}(V^{\sim})$ is a subset of $f^{-1}(V)$. Therefore, $f^{-1}(V)$ contains a $N_{g\beta g}$ OS $f^{-1}(V^{\sim})$ and thus, it is $N_{g\beta}$-open in X. Therefore, f is a $N_{g\beta}$ -continuous.

Remark 3.2

The subsequent example explains that the inverse of the above proposition does not true in general case.

Example 3.2

Suppose that $X = \{a, b, c\}$, $Y = \{1,2\}$, and let $f: X \longrightarrow Y$ be a function defined as $f(a) = 1$, $f(b) = 1$, & $f(c) = 2$.

Now, let us define a $N_{g\beta}$ X, defined as:

$g: [0,1] \longrightarrow X$, such that $g(t) = a$, for $0 \leq t < 0.5$, and $g(t) = b$, for $0.5 \leq t < 1$.

In this case, g is a $N_{g\beta}$ on X, and it is continuous in the case that the inverse image of any neutrosophic crisp open set in Y under the composition $f \circ g$ is a neutrosophic crisp open set in $[0,1]$. However, if we suppose that $B = \{2\}$, is a $N_{g\beta g}$ closed set in Y, then $f^{-1}(B) = \{c\}$, and B is not a $N_{g\beta g}$ closed set in X.

Example 3.3

Suppose that $f: \mathbb{R} \longrightarrow \mathbb{R}$, where $\mathbb{R}$ is the set of real numbers, let

$$f(x) = \begin{cases} x & if \quad x \leq 0 \\ 0 & if \quad x > 0 \end{cases}.$$

In this example if $x = 0$ so $f(x)$ is a discontinuous function, and if $x \leq 0$ or $x > 0$ then, $f(x)$ is the identity function and continuous. However, when x is greater than 0, $f(x)$ is a constant function with a value of 0, causing a jump discontinuity at $x = 0$.

Now, let define a $N_{g\beta}$ as a function $g(x)$ that is continuously varying from its lower bound to its upper bound.

In this case, let the function $g(x)$ defined as $g(x) = \begin{cases} -1 & for \quad x \leq 0 \\ 1 & for \quad x \geq 0 \end{cases}$.

In this example, $g(x)$ is a$N_{g\beta}$ function that is continuously varying from -1 to 1 as x increases from negative in finity to positive infinity. The function $f(x)$ is a continuous but does not a $N_{g\beta g}$-irresolute function.

Conclusions

In this paper, we present the concepts of neutrosophic β-open cover, neutronso- sophic β-compactness neutrosophic simply β-open cover, and neutrosophic simply

β-compactness in neutrosophic topological spaces. Also, we prove some properties and theorems on neutrosophic β-compactness and neutrosophic simply β-compac-

tness, and gives some remarks, examples.

Finally, we can extend the concepts of β -neutrosophic by connectednees.

References:

[1] Abd El Monsef M.E., (1980), "*Studied on Some Pretopological Concepts*", Ph. D. Thesis, Tanta Egypt.

[2] Abd El-Monsef M.E., El-Deeb S.N., and Mahmoud R.A., (1983) "*β-Open Sets and β-Continuous Mappings*", Bull. Fac. Sci. Assiut Univ. No. 12, PP.77–90.

[3]Andrijevic, D., (1996) "*On b-Open Sets*", Mat Vesn., Vol.48, PP.59-64.

[4]Arokiarani I., Dhavaseelan R., Jafari S., and Parimala M., (2017), "*On Some New Notions and Functions in Neutrosophic Topological Spaces*", Neutroso-phic Sets and Systems, Vol.16, PP.16-19, University of New Mexico.

[5]Arokiarani I., Dhavaseelan R., Jafari S., and Parimala M., (2017), "*On Some New Notions and Functions in Neutrosophic Topological Spaces*", Neutrosop-hic Sets and Systems, Vol.16, PP.16-19, University of New Mexico.

[6] Belal Nairat, (2018), "*On Some Properties of β- Open Sets*", International Journal of Applied Engineering Research ISSN 0973-4562 Volume 13, No. 6, pp. 3670-3672.

[7] Dhavaseelan R. and Jafari S., (2017), "*Generalized Neutrosophic Closed Sets*" New Trends in Neutrosophic Theory and Application, No.2 PP.261-273.

[8] Dhavaseelan R., Narmada Devi R., Jafari S., and Qays Hatem Imran, (2019), "*Neutrosophic α^m-continuity"*, Neutrosophic Sets and Systems, No. 27, PP.171-179.

[9] Hanif Page Md., and Qays Hatem Imran, (2020), "*Neutrosophic Generalized Homeomorphism*", Neutrosophic Sets and Systems, Vol.35, University of New Mexico.

[10]Levine N., (1963) "*Semi-Open Sets and Semi-Continuity in Topological Spaces*", Amer. Math Monthly, Vol.70, PP.36-41.

[11] Lipschutz S., (1986), "*Theory and Problems of General Topology*", Schums Series.

[12] Mashhuor, A. S., Abd El-Monsef M.F., and El-Deeb, S. N., (1982) "*On Precontinuous and Weak Precontinuous Mapping*", Proc. Math. Phys. Soc. Egypt, Vol.53, PP.47-53.

[13] Navalagi G.B., "*On Completely α- Irresolute Functions*", Topology Atlas Preprint.

[14] Noiri T., (1999), "*Almost αg-Closed Function and Separation Axioms*" Acta Math. Hungarica No. 3(82), PP.193-205.

[15] Parimala M., Jeevitha R., Smarandache F., Jafari S., and Udhayakumar R., (2018), "*Neutrosophic $\alpha\Psi$ Homeomorphim in Neutrosophic Topological Spaces Information"*, No. 9(187), PP.1-10.

[16] Pushpalatha A., and Nandhini T., (2019), "*Generalized Closed Sets Via Neutrosophic Topological Space*", Malaya Journal of Matematik, No.7(1), PP.50-54.

[17] Qays Hatem Imran, Dhavaseelan R., Ali Hussen Mohmood Al-obaidi, and Hanif Page Md., (2020), "*On Neutrosophic Generalized Alpha Generalized Continuity*", Neutrosophic Sets and Systems, Vol.35, University of New Mexico.

[18] Salma A.A. and Alblowi S.A., (2012) "*Neutrosophic Set and Neutrosophic Topological Spaces*", IOSR J. Math. 3:4, PP.31–35.

[19] Salama A.A., Smarandache F.,and Valeri K., (2014), "*Neutrosophic Closed Set and Neutrosophic Continuous Function*", Neutrosophic Sets and Systems, No. 4, PP.4-8.

[20]Salama A.A., Smarandache F. and Kroumov V., (2014), "*Neutrosophic Closed Set and Neutrosophic Continuous Function*", Neutrosophic Sets and Systems, No. 4, PP.4-8.

[21] Smarandache F., (2005). "*Neutrosophic Set - A generalisation of the Intuiti-onistic Fuzzy Sets*", Intern. J. Pure Appl. Math. 24:3, PP.287–297.

[22] Stone M.H., (1937), "*Application of the Theory of Boolean Rings to General Topology*", TAMS. No.41, PP.375-381.